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411 (mathematical method for optics)

$a_{-1} = \mathrm{Res}_{z=z_k} f(z) = \lim_{z\to z_k} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z-z_k)^m f(z))$
$= \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ (ϕ non zero at z_0.)

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tensor(rank)
tensors?, stress teasor.
outer product gives u tensors
rank: 0 - scalar, 1 - vector, 2 - $N \times Nmatrix$, ≥ 3 - tensor

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einstein notation
dot product: invariance to rotations. $\mathbf{a}\cdot\mathbf{b} = ||\mathbf{a}||||\mathbf{b}||\cos\theta$
einstein notation: sum over repeated indices is implicit: $\mathbf{A}\cdot\mathbf{B} = A_i B_i$
indices cannot appear more than twice within a term.
"free" indices
$|\mathbf{A}\times\mathbf{B}| = ||\mathbf{A}||||\mathbf{B}||\sin\theta$
$(\mathbf{A}\times\mathbf{B}) = \epsilon_{ijk}A_j B_k$
$\epsilon_{ijk} = \begin{cases} 0 & (i==j)\cup(j==k)\cup(k==i) \\ 1 & (1,2,3) \\ -1 & (3,2,1) \end{cases}$
$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$

$$\epsilon_{kij}\epsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$$

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$$

$\{\vec{A}\times(\vec{B}\times\vec{C})\} = \epsilon_{ijk}A_j\epsilon_{klm}B_lC_m = \epsilon_{kij}\epsilon_{klm}A_jB_lC_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})A_jB_lC_m = B_iA_jC_j - C_iA_jB_j = \{\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\}_i$
$\{\vec{A}\cdot(\vec{B}\times\vec{C})\}_i = A\epsilon_{ijk}B_jC_k = B_i\epsilon_{ijk}C_jA_k = \{\vec{B}(\vec{C}\times\vec{A})\}_i = C_i\epsilon_{ijk}A_jB_k = \{\vec{C}\cdot(\vec{A}\times\vec{B})\}_i$
$\{\vec{A}\otimes\vec{B}\}_{ij} = A_iB_j$
$\vec{A}\cdot\vec{B} = (A_x\ A_y\ A_z)\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$
outer product; $\vec{A}\otimes\vec{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}(B_x\ B_y\ B_z) = \begin{pmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{pmatrix}$
–
$\{\vec{r}\}_i = x_i$
$\{\nabla\}_i = \frac{\partial}{\partial x_i} \qquad (=\partial_i)$
$\{\nabla\times\vec{A}\}_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}A_k$
(Jacobian matrix) $\{\nabla\otimes\vec{A}\}_{ij} = \frac{\partial}{\partial x_i}A_j$
directional derivative: $(\vec{c}\cdot\nabla)\vec{A} = \vec{c}\cdot(\nabla\otimes\vec{A})$

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derivatives
$\phi(\vec{r})$ scalar function of $\vec{r}$
$\vec{A}(\vec{R}) = (A_x(x,y,z), A_y(x,y,z), A_z(x,y,z))$ (vector function of $\vec{r}$

meaningful derivatives - must have a meaning that doesn't depend on x,y,z.
these derivatives always has to do with vector differential operator "nabla" or "del"
Gradient (a vector)
$\nabla\phi$ = rate of changes of $\phi(r)$ points in the direction of maximum change
particles tends to goes to higher intensity via the gradient (when laser shine on the particle)

Divergence - local density of net flux
$\vec{A}(\vec{r})$
Net flux $= \int_S \vec{A}(\vec{r}) \cdot d\vec{\sigma}$
$Div[\vec{A}(\vec{r})]d\tau = \int_S \vec{A} \cdot d\vec{\sigma}$
$Div\{\vec{A}\} = \nabla \cdot \vec{A}(\vec{r})$
——— irf
Curl - local density of circulation
curl$\{\vec{A}\} = \nabla \times \vec{A}$

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second derivatives

name	vector notation	einstein notation
laplace	$\nabla \cdot (\nabla\phi)$	$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\phi = \nabla^2\phi$
	$\nabla \times (\nabla\phi)$	$\{\nabla \times (\nabla\phi)\}_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\phi = 0_i$
Hessian	$\nabla \otimes (\nabla\phi)$	$\{\nabla \otimes (\nabla\phi)\}_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\phi = \frac{\partial^2\phi}{\partial x_i \partial x_j}$
Grad-Div	$\nabla(\nabla \cdot \vec{A})$	$\{\nabla(\nabla \cdot \vec{A})\}_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_i$
Div.Curl	$\nabla \cdot (\nabla \times \vec{A})$	$\{\nabla \cdot (\nabla \times \vec{A})\} = \epsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_k = 0$
Curl.Curl = Grad.Div-Laplacian	$\nabla \times (\nabla \times \vec{A})$	$\{\nabla \times (\nabla \times \vec{A})\}_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}A_i = \{\nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}\}_i$
Laplacian acting on a vector	$\nabla \cdot (\nabla \otimes \vec{A})$	$\{\nabla \cdot (\nabla \otimes \vec{A})\}_i = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}A_i = \nabla^2\vec{A}$
		$\{\nabla \times (\nabla \otimes \vec{A})\} =$
		$\{\nabla \otimes (\nabla \otimes \vec{A})\} =$

$\nabla[\phi q] = q\nabla\phi + \phi\nabla q$
scalar times vector:
$\{\nabla \cdot (\phi\vec{A})\}_i = \frac{\partial}{\partial x_j}(\phi A_i) = \phi\frac{\partial A_i}{\partial x_i} + A_i\frac{\partial}{\partial x_i}\phi \qquad \Rightarrow \qquad \nabla \cdot (\phi\vec{A}) = \phi\nabla \cdot \vec{A} + \vec{A} \cdot \nabla\phi$
$\{\nabla \times (\phi\vec{A})\}_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi A_i) = \epsilon_{ijk}\phi\frac{\partial A_k}{\partial x_j} + \epsilon_{ijk}\frac{\partial\phi}{\partial x_j}A_k \qquad \Rightarrow \qquad \nabla \times (\phi\vec{A}) = \phi\nabla \times \vec{A} + (\nabla\phi) \times \vec{A}$
–
vector times vector:
$\{\nabla(\vec{A} \cdot \vec{B})\}_i = \frac{\partial}{\partial x_i}(A_j B)j) = A_j\frac{\partial}{\partial x_i}B_j + B_j\frac{\partial}{\partial x_i}A_j \qquad \Rightarrow \qquad \nabla(\vec{A} \cdot \vec{B}) = (\nabla \otimes \vec{B}) \cdot \vec{A} + (\nabla \otimes \vec{A}) \cdot \vec{B}$
$\{\nabla \cdot (\vec{A} \times \vec{B})\} = \frac{\partial}{\partial x_i}(\epsilon_{ijk}A_j B_k) = \epsilon_{ijk}B_k\frac{\partial}{\partial x_i}A_j + \epsilon_{ijk}A_j\frac{\partial}{\partial x_i}B_k = B \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
$\{\nabla \times (\vec{A} \times \vec{B})\}_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}A_l B_m = \epsilon_{kij}\epsilon_{klm}\left(B_m\frac{\partial}{\partial x_j}A_l + A_l\frac{\partial}{\partial x_j}B_m\right) = B_j\frac{\partial}{\partial x_j}A_i + A_i\frac{\partial}{\partial x_j}B_j -$
$B_i\frac{\partial}{\partial x_j}A_j - A_j\frac{\partial}{\partial x_j}B_i$
$\Rightarrow \nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B}$

Fundamental Theorem of calculus

(1D) $\displaystyle\int_a^b f'(x)dx = f(b) - f(a)$

Def Deriv. $f'(x)\Delta x = f(x+\Delta x) - f(x)$

Def Int. $\displaystyle\int_a^b F(x)dx = \lim_{\Delta x\to 0}\sum_{n=0}^{N-1} F(x_n)\Delta x \qquad (x_n = a + n\Delta x)$

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(3D)

$\nabla\phi\cdot\Delta\vec{r} = \phi(\vec{r}+\Delta\vec{r}) - \phi(\vec{r})$

Integral along path (line integral): $\displaystyle\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r})\cdot d\vec{\lambda} = \lim_{\Delta\vec{\lambda}_n\to\vec{0}}\sum_{n=1}^{N-1}\vec{F}(\vec{r}_n)\cdot\Delta\vec{\lambda}_n$

If $\vec{F}(\vec{r}) = \nabla\phi(\vec{r})$

$$\int_{\vec{r}_a}^{\vec{r}_b}\nabla\phi\cdot d\vec{\lambda} = \lim_{\Delta\vec{\lambda}_n\to\vec{0}}\sum_{n=0}^{N-1}\underbrace{\nabla\phi(\vec{r}_n)\cdot\Delta\vec{\lambda}_n}_{\phi(\vec{r}_{n+1})-\phi(\vec{r}_n)} = \phi(\vec{r}_b) - \phi(\vec{r}_a)$$

Def. of Div. $\displaystyle\nabla\cdot\vec{A}\Delta\tau = \int_{S\text{ of }\Delta\tau}\vec{A}\cdot d\vec{\sigma}$

$$\int_\tau f(\vec{r})d\tau = \lim_{\Delta\tau_n\to 0}\sum_n f(\vec{r}_n)\Delta\tau_n$$

if $f(\vec{r}) = \nabla\cdot\vec{A}(\vec{r})$

$$\int_\tau\nabla\cdot\vec{A}d\tau = \int_{S\text{ of }\Delta\tau}\vec{A}\cdot d\vec{r}$$

Def: $\displaystyle(\nabla\times\vec{A})\cdot\Delta\vec{\sigma} = \oint_{C\text{ of }\Delta\vec{\sigma}}\vec{A}\cdot d\vec{\lambda}$

$$\int_S(\nabla\times\vec{A})\cdot d\vec{\sigma} = \lim_{\Delta\vec{\sigma}_n\to\vec{0}}\sum_n\underbrace{\nabla\times\vec{A}(\vec{r}_n)\cdot\Delta\vec{\sigma}_n}_{\oint_{C\text{ of }\Delta\vec{\sigma}}\vec{A}\cdot d\vec{\lambda}} = \oint_{C\text{ of }\Delta\vec{\sigma}}\vec{A}\cdot d\vec{\lambda}$$

$$\int_{\vec{r}_a}^{\vec{r}_b}\nabla\phi\cdot d\vec{\lambda} = \phi(\vec{r}_b) - \phi(\vec{r}_a)$$

F. T. of Divs. (Gauss' Theorem) $\displaystyle\int_\tau\nabla\cdot\vec{A}d\tau = \int_{S\text{ of }\Delta\tau}\vec{A}\cdot d\vec{r}$

F.T. of Curls (Stokes' Theorem) $\displaystyle\int_S(\nabla\times\vec{A})\cdot d\vec{\sigma} = \oint_{C\text{ of }\Delta\vec{\sigma}}\vec{A}\cdot d\vec{\lambda}$

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functions of complex variables

complex functions of complex varibles

$f(z) = u(x,y) + iv(x,y)$

derivative

$dz = dx + idy$

$f'(z)dz = Re\{f'\}dx - Im\{f'\}dy + i(Im\{f'\}dx + Re\{f'\}dy)$

RHS $f(z+dz)-f(z) = u(x+dx,y+dy)-u(x,yy)+i[v(x+dx,y+dy)-v(x,y)] = dxu_x+dyu_y+i[dxv_x+dyv_y]$

LHS = RHS $\forall dxdy$

$u_x = v_y, u_y = -v_x$ Cauchy-Riemann conditions.

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existence of $f'(z_0)$ requires $u_x = v_y, u_y = -v_x$ at point z_0.

Write

$$u_r = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, u_\theta = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta} \qquad \text{and for v too}$$

its called Cauchy-Riemann equations, but equality doesn't imply existence of $f'(z_0)$.

first order partial derivative of u, v with respect to x, y (r,θ) exist everywhere in the neighborhood, and are continuous at (x_0, y_0) and satisfy Cauchy-Riemann equations

$$u_x = v_y, u_y = -v_x \qquad ru_r = v_\theta, u_\theta = -rv_r$$

at that point.
Then $f'(z_0)$ exist, its value being $f'(z_0) = u_x + iv_x = e^{-i\theta}(u_r + iv_r)$

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analytic functions.
analytic in an open set S if derivative everywhere in that set.
analytic at a point z_0 if it is analytic in some neighborhood of z_0.
entire function(holomorphic) - function that is analytic at each point in the entire plane.
analytic in domain D requires continuity, first order partial derivative of u, v with respect to x, y (r, θ) exist everywhere in the neighborhood, and are continuous at (x_0, y_0) and satisfy Cauchy-Riemann equations.
if two functions are analytic in a domain D, their sum and product are both analytic in D., quotient too (when denominator $\neq 0$.
composition of 2 analytic functions is analytic
singular point z_0 - fails to be analytic at that point, but is analytic at some point in every neighborhood of zzz_0
an analytic function from a well behaved function of a real variable x by replacing x with z

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integral
integrate counterclockwise $\oint_C f(z)dz = \oint_C (u+iv)(dx+idy) = \oint_C (udx - vdy) + i\oint_C (vdx + udy) = \oint_C \vec{A} \cdot d\vec{\lambda} + i\oint_C \vec{B} \cdot d\vec{\lambda}$
Stoke's theorem $\rightarrow \int_{S \text{ inside } C} \nabla \times \vec{A} \cdot d\vec{\sigma} + i\int_{S \text{ inside } C} \nabla \times \vec{B} \cdot d\vec{\sigma} = \int_{S \text{ in } C} \{\nabla \times \vec{A}\}_z dxdy + i\int_{S \text{ in } C} \{\nabla \times \vec{B}\}_z dxdy = \int_{S \text{ in } C} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\int_{S \text{ in } C} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$
$= 0$ due to C-R conditions

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f(z) ana
$\int_C \frac{f(z)}{z-z_0}dz$. Make C a circle around z_0 as small as possible. $\int_C \frac{f(z)}{z-z_0}dz = \int_0^{2\pi} if(z)d\theta = 2\pi i f(z)$

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function f and its conjugate are both analytic $\rightarrow f(z) = c_0 + ic_1$.
–
Harmonic: in given domain, has continuous partial derivatives of the $1^{st}, 2^{nd}$ order and satisfy Laplace's eq.
Second way to prove harmonic: $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u, v are harmonic in D.
Laplace's eq. $u_{xx} + u_{yy} = 0$, function v too.
$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0$, function v too.
–
Reflection principle
$\overline{f(z)} = f(\overline{z})$, On real axis $f(z) = \overline{f(\overline{z})}$
–
$\frac{d}{dz}e^z = e^z$ everywhere in the z plane, e^z is entire.
periodic by $2\pi i$.
e^z can be negative!.
output of e^z can be any non-zero complex number.
–
$\log z = \text{Log } z + 2n\pi i \qquad \text{Log } z = \ln r + i\Theta$
Notice, the function is log, not ln! (dont make sense but it's defined this way)
$\frac{d}{dz}\log z = \frac{1}{z} \qquad z \neq 0, \alpha < arg\ z < \alpha + 2\pi$

For Log z, Arg z $\in \{-\pi, \pi\}$, domain slightly diff.
$\log(z_1 z_2) = \log z_1 + \log z_2$
$arg(z_1 z_2) = arg z_1 + arg z_2$
–
$z \neq 0$, c any complex number. power function
$z^c = e^{c \log z} \qquad z \neq 0.$
can be multiple-valued.
$\frac{d}{dz} z^c = c z^{c-1}$
$\frac{d}{dz} c^z = c^z \log c$, or $\frac{d}{dc} z^c = z^c \log z$.
–
Zeros of sin, cos are the zeros of real line.
$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}$
$\sinh y = \frac{e^{y} - e^{-y}}{2}, \cosh y = \frac{e^{y} + e^{-y}}{2}$
$\cosh^2 y - \sinh y^2 = 1$
$\sin z = \sin x \cosh y + i \cos x \sinh y$
$\cos z = \cos \cosh y - i \sin x \sinh y$
where $z = x + iy$
$\sin z, \cos z$ not bounded on complex plane.
$|\sin z|^2 = \sin^2 x + \sinh^2 y$
$|\cos z|^2 = \cos^2 x + \sinh^2 y$
–
$\frac{d}{dz} \sinh z = \cosh z, \frac{d}{dz} \cosh z = \sinh z$
$-i \sinh(iz) = \sin z, \cosh(iz) = \cos z$
$-i \sin(iz) = \sinh z, \cos(iz) = \cosh z$

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zeros of sinh at $i(n\pi)$, zeros of cosh at $i(n\pi + \frac{\pi}{2})$.
$-i \sinh(iz) = \sin z, \cosh(iz) = \cos z$
hyperbolic functions periodic with period $2\pi i$.
$\sinh(-z) = -\sinh z, \cosh(-z) = \cosh z$
$\cosh^2 z - \sinh^2 z = 1$
$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$
$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$
$\sinh = \sinh x \cos y + i \cosh x \sin y$
$\cosh z = \cosh x \cos y + i \sinh x \sin y$
$|\sinh z|^2 = \sinh^2 x + \sin^2 y$
$|\cosh z|^2 = \sinh^2 x + \cos^2 y$
$\sinh z = 0$ iff $z = n\pi i$
$\cosh z = 00$ iff $z = \left(\frac{\pi}{2} + n\pi\right) i$
$\frac{d}{dz} \tanh z = \operatorname{sech}^2 z, \frac{d}{dz} \coth z = -\operatorname{csch}^2 z$
$\frac{d}{dz} \operatorname{sech} z = -\operatorname{sech} z \tanh z, \frac{d}{dz} \operatorname{csch} z = -\operatorname{csch} z \coth z$
–
inverse tri, hyperbolic
$\sin^{-1} z = -i \log[iz + \sqrt{1 - z^2}]$
$\cos^{-1} z = -i \log[z + i\sqrt{1 - z^2}]$
$\tan^{-1} z = \frac{i}{2} \log \frac{i + z}{i - z}$
they are multiple-valued
$\frac{d}{dz} \sin^{-1} z = \frac{1}{\sqrt{1 - z^2}}$

$\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1-z^2}}$
$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$
$\sinh^{-1} z = \log[z + \sqrt{z^2+1}]$
$\cosh^{-1} z = \log[z + \sqrt{z^2-1}]$
$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$

$$w'(t) = u'(t) + iv'(t) \tag{1}$$

$\frac{d}{dt}w(t)^2 = 2w(t)w'(t)$ (by 1)
$\frac{d}{dt}e^{z_0 t} = z_0 e^{z_0 t}$ (by 1)
$\star$ mean value theorem no longer apply.
$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$
existence of integral ensured if functions are piecewise continuous.
mean value theorem for integral doent apply either.
–
Arc - is simple if it does not cross itself.
simple closed curve, Jordan curve. Positively oriented if in counterclockwise direction.
contour - piecewise smooth arc.
–
line integral
$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$
$\int_C z_0 f(z)dz = z_0\int_C f(z)dz$
–
$\left|\int_a^b w(t)dt\right| \le \int_a^b |w(t)|dt$
$|f(z)| \le M \Rightarrow \left|\int_C f(z)dz\right| \le ML$
–
Cauchy-Goursat theorem
if f analytic at all points interior 2 and on a simple closed contour C, then $\int_C f(z)dz = 0$. converse also true.
analytic throughout a simply connected domain, then true for every closed contour C lying in D. f has antiderivative everywhere in D.
–
analytic on and within C, z_0 any point interior to C.
$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0}$
$\int_C \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0)$
$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \qquad n \in \mathbb{N}_0$
$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}} \qquad n \in \mathbb{N}_0$
f analytic at $z_0 \Rightarrow$ its derivatives of all orders are analytic at z_0 too
$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \qquad n \in \mathbb{N}$ where M_R = maximum value of $|f(z)|$ on C_R

–
Liouville's theorem
If a function f is entire and bounded in complex place, then $f(z)$ is constant throughout the plane.
–
fundamental theorem of algebra
Any polynomial with degree $n \leq 1$ has at least 1 zero.
–
If f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D.
–
Taylor series (complex)
$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ (over $n!$)
negative power of $(z - z_0)$
e^z, compose with $(z + n)$, multiply by by e^n and taylor expand.
–
Laurent's theorem
f analytic on annular domain, C (+) closed contour in that domain. Then at each point on the domain,

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \\
a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}} \\
b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \\
f(z) &= \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \\
c_n &= \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}
\end{aligned}
$$

–
uniform convergence of Taylor series - all points on the largest circle centered at z_0.
–
multiplication of power series
$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z) \qquad n \in \mathbb{N}$
$c_n = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k}$
–
residue
singular point z_0 is isolated if these is a deleted ϵ neighborhood of z_0 throughout which f is analyic.
$\int_C f(z)dz = 2\pi i b_1 = 2\pi i \text{Res}_{z=z_0} f(z)$
$\text{Res}_{z=z_0} f(z)$ is power series in functions of $z - z_0$
Partial fraction can help
point at infinity is said to be an isolated singular point of f.
$\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} \left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)$
–
Removable (all negative power of $z - z_0$ are 0), essential (infinitely many nonzero negative power of $z - z_0$) , or a pole (finite nonzero negative power of $z - z_0$, largest negative power m means pole of order m).
Picard's theorem - each neighborhood of an essential singular point, a function assumes every finite value,

with one possible exception, an infinite number of times.
–
residue at poles
z_0 is a pole of order m of f. equivalent to
$f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \qquad m \in \mathbb{N}$$

where $\phi(z)$ is analytic and nonzero at z_0.
$\text{Res}_{z=z_0} f(z) = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$
–
zero of order m at z_0: $f(z_0) = ... = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0$
if $f(z) = \dfrac{p(z)}{q(z)}$ where singularity at z_0, if $p(z_0) \neq 0, q(z_0) = 0, q'(z_0) \neq 0$, then $\text{Res}_{z=z_0} f = \dfrac{p(z_0)}{q'(z_0)}$
–
if f analytic at z_0, $f(z_0) = 0$ but $f(z)$ is not identically equal 2 zero in any neighborhood of z_0.
Then $f(z) \neq 0$ throughout some deleted neighborhood of z_0.
–
if f analytic throughout neighborhood N_0 of z_0, $f(z_0) = 0$ at every point in domain D or line segment L containing z_0, then $f(z) \equiv 0$ in N_0
Then $f(z) \neq 0$ throughout some deleted neighborhood of z_0.
–
if z_0 a removable singular point of f, then f is bounded and analytic in some deleted neigh ϵ of z_0.
f bounded, analytic in deleted ϵ neigh of z_0. If f is not analytic at z_0, then it has a removable singularity there.
essential singular point: in each deleted neigh of the ESP, f assumes values arbitrarily close to any given number.
if z_0 is a pole, then $\lim\limits_{z \to z_0} f(z) = \infty$.
–
$\displaystyle\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$
C_R semi-circular arc, z_k are singular points inside the semi circle.
Cauchy principal value (P.V.)
P.V. $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} f(x)dx$
The improper integral doesnt always converges even if its P.V. exists.
–
For a simple pole on x-axis, $\displaystyle\lim_{\rho \to 0} \int_{C_\rho} f(z)dz = -B_0 \pi i$
if all singularity on one side of the x-axis, we can make a contour with two concentric circles connected by a line along the x-axis.
–
definite integral with sin,cos
$\displaystyle\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta = \int_C F\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right)\frac{dz}{iz}$
–
argument principle
meromorphic in domain D if analytic throughout except poles.
if f meromorphic in D interior to C, analytic and nonzero on C, then winding number

$$\Gamma = \frac{1}{2\pi}\Delta_C arg\ f(z) = Z - P$$

where Z is the number of zero, P number of poles of f inside C, counting multiplicities.

–

rouche's theorem

f, g analytic inside, and on C, $|f(z)| > |g(z)|$ at each point on C.

Then $f(z), f(z) + g(z)$ have the same number of zeros, counting multiplicities inside C

---

branch point: points that cause that function to be multi-valued. $f(z)$ can be finite/infinity at branch points. At branch points, if we close an arbitrarily small loop around that point, it travels and end up at a different value.

branch cut: $\sqrt{z}$ has 2 Riemann sheets. The whole thing is a Riemann surface.

if taylor expand $\sqrt{z}$ at $z_0 > 0$, uniform convergence in the circle of radius z_0 centered at z_0

Product of fractional powers.

$f(z) = \sqrt{z^2 - 1} = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}$

branch point at ± 1

branch cut: $(-1, 1)$ or $(-\infty, -1) \cup (1, \infty)$

In general $f(z) = (z - z_0)^{\frac{n}{m}}$ where the fraction is irreducible. function has m riemann sheets

$f(z) = (z - z_0)^{\alpha}(z - z_2)^{\beta}(z - z_3)^{\gamma}$ # riemann sheets is lowest common denuminator of α, β, γ

if $\alpha + \beta + \gamma = \mathbb{Z}$ - can join the branch points with finite branch cut.

sums: $f(z) = \sqrt{z-1} + \sqrt{z+1}$ has 4 Riemann sheets.

# of Riemann sheets is the product of the number of Riemann sheets for each. term

---

Poles: points where function diverge but doesn't cause multivalue.

$f(z)$ has finitely many negative powers of $(z - z_0)^m$.

Essential singularity: infinitely many negative powers of $(z - z_0)^m$

Laurent Series

Starting from CIF $f(z) = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z')}{z' - z} dz'$

f analytic inside and along C.

connected contour of different circle with r, R .

if all singularities inside C_r are not branch points or if they include branch pts, but these can be joined by a finite branch cuts. Then the contour integral is just the two circle line integrals (closed)).

(and opposite direction)

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{C_R} \frac{f(z')}{z'-z}dz' - \frac{1}{2\pi i}\oint_{C_r} \frac{f(z')}{z'-z}dz' &= \frac{1}{2\pi i}\oint_{C_R} \frac{f(z')}{(z'-z_0)-(z-z_0)}dz' - \frac{1}{2\pi i}\oint_{C_r} \frac{f(z')}{(z'-z_0)-(z-z_0)}dz' \\
&= \frac{1}{2\pi i}\oint_{C_R} \frac{f(z')}{z'-z_0}\frac{1}{1-\frac{z-z_0}{z'-z_0}}dz' + \frac{1}{2\pi i}\oint_{C_r} \frac{f(z')}{z'-z_0}\frac{1}{1-\frac{z-z_0}{z'-z_0}}dz' \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\oint_{C_R} \frac{f(z')}{z'-z_0}\left(\frac{z-z_0}{z'-z_0}\right)^n dz' + \sum_{n=0}^{\infty} \frac{1}{2\pi i}\oint_{C_r} \frac{f(z')}{z'-z_0}\left(\frac{z-z_0}{z'-z_0}\right)^n dz' \\
&= \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i}\oint_{C_R} \frac{f(z')}{(z'-z_0)^{n+1}}dz'}_{a_n}(z-z_0)^n + \sum_{m=0}^{\infty} \underbrace{\frac{1}{2\pi i}\oint_{C_R} \frac{f(z')}{(z'-z_0)^{n+1}}dz'}_{a_{-m-1}=a_n}(z-z_0)^{-m-1} \\
&= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=-\infty}^{-1} a_n (z-z_0)^n \\
&= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \\
a_n &= \frac{1}{2\pi i}\int_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}
\end{aligned}
$$

we can find a LS around a pole.
Taylor series on the region of circle centered at origin with radius reaching first singularity.
Laurent series on other regions (but if there's branch points, laurent series may not be valid).
remember technique: $|z-2|>1, f(z)=\dfrac{1}{z-2}\dfrac{1}{1+\frac{1}{z-2}}$
remember technique: when z_0 is in middle between singularity, partial fraction is not needed.
$f(z)=\sqrt{z^4-4}$ expand at $z_0=0$
$|z|<2, f(z)=\pm i\sqrt{1-\dfrac{z^2}{4}}$
Use binomial expansion $(1+x)^{\frac{1}{2}}=..$

$\displaystyle\int_C f(z)dz=2\pi i b_1=2\pi i \mathrm{Res}_{z=z_0}f(z)$
$\mathrm{Res}_{z=z_0}f(z)$ is power series in functions of $z-z_0$
Partial fraction can help
point at infinity is said to be an isolated singular point of f.
$\mathrm{Res}_{z=\infty}f(z)=-\mathrm{Res}_{z=0}\left(\dfrac{1}{z^2}f\left(\dfrac{1}{z}\right)\right)$
If all singularities inside C are poles (not branch points), then the "bridge" subcontours cancel.
$a_{-1}=\mathrm{Res}_{z=z_k}f(z)=\lim\limits_{z\to z_k}\dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left((z-z_k)^m f(z)\right)$
$\mathrm{Res}_{z=z_k}f(z)=\lim\limits_{z\to z_k}(z-z_0)f(z)$

---

3 types of integrals
$I_1=\displaystyle\int_{-\infty}^{\infty}f(x)dx$
Do this type of integral with semi circle with $R\to\infty$ contour lying on x-axis.
$\displaystyle\oint_{C_R}f(z)dz=I_1+i\int_0^{\pi}\lim_{R\to\infty}zf(z)d\theta$
if $f(z)\to 0$ faster than $\dfrac{1}{z}$ as $z\to\infty$
then $I_1=\displaystyle\oint_{C_R}f(z)dz=2\pi i\sum_{k\text{ above x-axis}}\mathrm{Res}\{f(z),z_k\}=-2\pi i\sum_{k\text{ below x-axis}}\mathrm{Res}\{f(z),z_k\}$

---

example:
$\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2}=2\pi i\lim_{z\to ia}\frac{d}{dz}\left(\frac{(z-ia)^2}{(z^2+a^2)^2}\right)=-4\pi i(z+ia)^{-3}|_{ia}=\frac{\pi}{2a^3}$

$I_2=\displaystyle\int_{-\infty}^{\infty}f(x)e^{ikx}dx \qquad k\in\mathbb{R}$
for $e^{ikz}=e^{ikx-ky}$ For $k>0$, use semicircle above x-axis.
For $k<0$, use semicircle below x-axis.
exmaple: $k>0, \displaystyle\oint_{C_R}f(z)e^{ikz}dz=I_2-\lim_{R\to\infty}\int_0^{2\pi}\star d\theta$
Second term $\to 0$ if $M(R)\to 0$ as $R\to\infty$. ($f(z)\to 0$ as $|z|\to\infty$)
$k>0: \qquad 2\pi i\mathrm{Res}\left\{f(z)e^{ikz},\forall z_0\right\}=I \qquad$ (upper semi-circle)
$k<0: \qquad -2\pi i\mathrm{Res}\left\{f(z)e^{ikz},\forall z_0\right\}=I$ (lower semi-circle, negative sign cuz contour counterclockwise)

$I=\displaystyle\int_0^{\infty}\frac{\sqrt{x}}{x^2+a^2}dx \qquad a>0$
Contour integral: $\displaystyle\lim_{\epsilon\to 0,R\to\infty}\int_{\epsilon}^{R}\frac{r^{\frac{1}{2}}dr}{r^2+a^2}+\lim_{R\to\infty}\int_0^{2\pi}\frac{R^{\frac{1}{2}}e^{\frac{i\theta}{2}}}{R^2e^{i2\theta}+a^2}iRe^{i\theta}d\theta+\lim_{\epsilon\to 0,R\to\infty}\int_R^{\epsilon}\frac{(re^{i2\pi})^{\frac{1}{2}}}{r^2e^{i4\pi}+a^2}e^{i2\pi}dr+$
$\displaystyle\lim_{\epsilon\to 0}\int_{2\pi}^{0}\frac{\epsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}}{\epsilon^2e^{i4\pi}+a^2}\epsilon e^{i\theta}d\theta$

$2I \;=\; \dfrac{1}{2}\oint_C \dfrac{z^{\frac{1}{2}}}{z^2+a^2}dz \;=\; \pi i \text{Res}\left\{\dfrac{z^{\frac{1}{2}}}{z^2+a^2}, ae^{i\frac{\pi}{2}}, ae^{i\frac{3\pi}{2}}\right\} \;=\; \pi i \sum \left.\dfrac{z^{\frac{1}{2}}}{2z}\right|_{ae^{i\frac{\pi}{2}}, ae^{i\frac{3\pi}{2}}} \;=\; \pi i\left(\dfrac{e^{-i\frac{\pi}{4}}}{2\sqrt{a}} + \dfrac{e^{-i\frac{3\pi}{4}}}{2\sqrt{a}}\right) \;=$
$\dfrac{\pi i}{\sqrt{2a}}$
$\displaystyle\int_0^\infty \frac{x^\alpha}{(x^n+a^n)^N(x^m+b^m)^M}dx$
if $n=m$
or if we just have $\displaystyle\int_0^\infty \frac{x^\alpha}{(x^n+a^n)^N}dx$
pizza slice containing first singularity.

---

example
$I = \displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2+a^2}dx$
contour integral, upper semi circle, with bump at the origin.
$\displaystyle\lim_{\epsilon\to 0, R\to\infty}\int_\epsilon^R \frac{r^{\frac{1}{2}}dr}{r^2+a^2} - 0 + \lim_{\epsilon\to 0, R\to\infty}\int_R^\epsilon \frac{r^{\frac{1}{2}}e^{i\frac{\pi}{2}}dre^{i\pi}}{r^2e^{i2\pi}+a^2}$
$= (1+e^{i\frac{\pi}{2}})I = 2\pi i \text{Res}\left\{\dfrac{z^{\frac{1}{2}}}{z^2+a^2}, ae^{i\frac{\pi}{2}}\right\} = \pi i z^{-\frac{1}{2}}|_{ae^{i\frac{\pi}{2}}} = \dfrac{\pi i}{\sqrt{a}}e^{-\frac{\pi}{4}} = \dfrac{\pi}{\sqrt{a}}e^{\frac{\pi}{4}}$
LHS is $2\cos\left(\dfrac{\pi}{4}\right)e^{i\frac{\pi}{4}}I \qquad \Rightarrow \qquad I = \dfrac{\pi}{\sqrt{2a}}$
$\displaystyle\oint_C \ln z f(z)dz = \lim_{\epsilon\to 0, R\to\infty}\int_\epsilon^R \ln r f(r)dr + \lim_{R\to\infty}\int_0^{2\pi} ln(Re^{i\theta})f(Re^{i\theta})iRe^{i\theta}d\theta + \lim_{\epsilon\to 0, R\to\infty}\int_R^\epsilon \ln(re^{i2\pi})f(re^{i2\pi})dre^{i2\pi} +$
$\displaystyle\lim_{\epsilon\to 0}\int_{2\pi}^0 \ln(\epsilon e^{i\theta})f(\epsilon e^{i\theta})i\epsilon e^{i\theta}d\theta$
2^{nd} term goes $\to 0$ if $f.t. \dfrac{1}{z^{1+\delta}}$ with $\delta > 0$ as $|z|\to\infty$ ($f(z)$ goes $\to 0$ faster than $\dfrac{1}{z^{1+\delta}}$ as $|z|\to\infty$)
4^{th} term goes $\to 0$ if $f(0)$ is well defined .
LHS becomes $-2\pi i I = 2\pi i \displaystyle\sum_n \text{Res}\left\{lnzf(z), z_n\right\} \qquad \theta\in(0,2\pi)$

---

example
$I = \displaystyle\int_0^\infty \frac{dx}{x^3+a^3} \qquad a>0$, integral real, positive, $\propto a^{-2}$
contour integral of $\ln z f(z)$ with connected CC C_R, clockwise C_ϵ
$\displaystyle\lim_{\epsilon\to 0, R\to\infty}\int_\epsilon^R \frac{\ln r}{r^3+a^3}dr + \lim_{\epsilon\to 0, R\to\infty}\int_R^\epsilon \frac{\ln r - i2\pi}{r^3e^{i2\pi}+a^3}dre^{i2\pi}$
$-2\pi i \qquad \Rightarrow \qquad I = -\sum \text{Res}\left\{\dfrac{\ln z}{z^3+a^3}; ae^{i\frac{\pi}{3}}, ae^{i2\pi}, ae^{i\frac{5\pi}{3}}\right\} = -\sum \left.\dfrac{\ln z}{3z^2}\right|_{ae^{i\frac{\pi}{3}}, ae^{i2\pi}, ae^{i\frac{5\pi}{3}}}$
$= -\left(\dfrac{lna + i\frac{\pi}{2}}{3a^2e^{i\frac{2\pi}{3}}} + \dfrac{\ln a + i\pi}{3a^2e^{i2\pi}} + \dfrac{\ln a + i\frac{5\pi}{3}}{3a^2e^{i\frac{5\pi}{3}}}\right) = \dfrac{2\pi}{3\sqrt{3}a^2}$

---

definite integral with sin,cos, C is unit circle
$\displaystyle\int_0^{2\pi} F(\sin\theta,\cos\theta)d\theta = \int_C F\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right)\frac{dz}{iz}$

$I = \displaystyle\int_{-\infty}^\infty \frac{\sin(kx)}{x}dx = \frac{1}{2i}\left(\int_{-\infty}^\infty \frac{e^{ikx}}{x}dx - \int_{-\infty}^\infty \frac{e^{-ikx}}{x}dx\right)$
$k>0 \qquad I = \dfrac{1}{2i}\left(0 + 2\pi i \text{Res}\left\{\dfrac{e^{-ikz}}{z}; 0\right\}\right) = \pi$
$k<0 \qquad I = \dfrac{1}{2i}\left(-2\pi i \text{Res}\left\{\dfrac{e^{ikz}}{z}; 0\right\}\right) = -\pi$
$k=0 \qquad \Rightarrow \qquad I = 0, I = \pi\text{sgn}(k)$

$$\int_{-a}^{b}\frac{1}{x}dx = \lim_{\epsilon_1\to 0}\int_{-a}^{-\epsilon 1}\frac{dx}{x} + \lim_{\epsilon_2\to 0}\int_{\epsilon_2}^{b}\frac{dx}{x} = \lim_{\epsilon_1,\epsilon_2\to 0}\ln\frac{b\epsilon_1}{a\epsilon_2} = \text{anything real depending on how } \epsilon_1,\epsilon_2\to 0$$

integral not well define (simple pole)

We can define the "Principal value" integral $\mathcal{P}\int_{-a}^{b}\frac{dx}{x} = \lim_{\epsilon\to 0}\left(\int_{-a}^{-\epsilon}\frac{dx}{x} + \int_{\epsilon}^{b}\frac{dx}{x}\right) = \ln\frac{b}{a}$

1 simple pole on contour: $\mathcal{P}\oint_C f(z)dz =$ difference of 2 contour integral (leap over pole by semi circle. $z = z_0 + \epsilon e^{i\theta}$

Laurentz series for the semi circle integral.

$$\int_{C_\epsilon} f(z)dz = \lim_{\epsilon\to 0}\int_{\theta_0+\pi}^{\theta_0}\left(\frac{a_{-1}}{\epsilon e^{i\theta}} + a_0 + a_1\epsilon e^{i\theta} + ...\right) i\epsilon e^{i\theta}d\theta$$

$$= \int_{\theta_0+\pi}^{\theta_0} ia_{-1}d\theta = -\pi i \text{Res}\{f(z), z_0\}$$

$$\mathcal{P}\oint_C f(z)dz = 2\pi i \sum_{n \text{ inside } C}\text{Res}\{f(z), z_n\} + \pi i \sum_{n \text{ along } C}\text{Res}\{f(z), z_n\}$$

If simple pole is along the corner of contour of $90°$, then it only catch $1/4$ of the residue.

Integral	Use	Contour	Condition
$\int_{-\infty}^{\infty} f(x)dx$	$\oint_C f(z)dz$	$\uparrow C_R, \downarrow C_R$	$zf(z)\to 0$ as $R\to\infty$
$\int_{-\infty}^{\infty} f(x)e^{ikx}dx$	$\oint_C f(z)e^{ikz}dzz$	$k>0 :\uparrow C_R, f<0 :\downarrow C_R$	$f(z)\to 0$ as $\lvert z\rvert\to\infty$
$\int_{0}^{\infty} x^\alpha f(x)dx$	$\oint_C z^\alpha f(z)dz$	pacman	$z^\alpha f(z)\to 0$ faster thanz^{-1} as $\lvert z\rvert\to\infty$,
$\int_{0}^{\infty} f(x)dx$	$\oint_C \ln z f(z)dz$	pacman	$f(z)\to 0$ faster than $z^{-1-\delta}$, f
$\int_{0}^{2\pi} F(\cos\theta,\sin\theta)d\theta$	$\oint_{UC} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$	UC	none

Residue theory to solve infite sums

$S = \sum_{n=-\infty}^{\infty} f(n) = \oint_C f(z)g(z)dz$ where g has simple poles at $\mathbb{Z}$

try $g(z) = \frac{\pi\cos(\pi z)}{\sin(\pi z)} = \pi\cot(\pi z)$

$\text{Res}\{g(z), n\} = 1$

Contour: square: $C_n : y = \pm\left(n+\frac{1}{2}\right), x = \pm\left(n+\frac{1}{2}\right) \lim_{n\to\infty}\oint_{C_n} f(z)g(z)dz$

Contour integral $= 0$ if $zf(z)\to\infty$ as $|z|\to\infty$

Then $0 = \sum_{\text{poles of f}}\text{Res}\{f(z)g(z), z_k\} + \sum_{n=-\infty}^{\infty}\underbrace{\text{Res}\{f(z)g(z); n\}}_{f(n)}$

$\therefore S = -\sum_{\text{poles of f}}\text{Res}\{f(z)\pi\cot(\pi z), z_k\} \qquad \lim_{|z|\to\infty} zf(z) = 0$

$S = \sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} \qquad a>0$

$S = -\text{Res}\left\{\frac{1}{z^2+a^2}\pi\cot(\pi z); \pm ia\right\} = \frac{\pi}{a}\coth(\pi a)$

Gaussian integrals

$I_0 = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}}dx$

$I_0^2 = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{ay^2}{2}} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{a(x^2+y^2)}{2}} dx$

change to polars: $= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{ar^2}{2}} r dr d\theta = 2\pi \int_0^{\infty} e^{-\frac{ar^2}{2}} r dr = 2\pi \int_0^{\infty} e^{-au} du = 2\pi \left. \frac{e^{-au}}{-a} \right|_0^{\infty}$

if $\mathrm{Re}\{a\} > 0$, then $I_0^2 = \frac{2\pi}{a}$

$I = \sqrt{\frac{2\pi}{a}}$

$I_n = \int_{-\infty}^{\infty} x^n e^{-\frac{ax^2}{2}} dx$ n is nonnegative integer

$I_{n=odd} = 0$

integration by parts: $dv = xe^{-\frac{ax^2}{2}}dx, u = x^{n-1}$

$= -\frac{e^{-\frac{ax^2}{2}}}{a} x^{n-1} \Big|_{-\infty}^{\infty} + \frac{n-1}{a}\int_{-\infty}^{\infty} x^{n-2} e^{-\frac{ax^2}{2}} dx$

$= \frac{n-1}{a} I_{n-2} = \frac{(n-1)!!}{a^{\frac{n}{2}}}$

$\sqrt{\frac{2\pi}{a}}$

$n!! = n(n-2)(n-4)...$

second method: $\frac{d}{da}$

($\star$) remember techniques: derivative, recursion.

---

if $a = -ib$

$I_0 = \int_{-\infty}^{\infty} e^{\frac{ibx^2}{2}} dxx = 2\int_0^{\infty} e^{\frac{ibx^2}{2}} dx = 2\left(\int_0^{\infty} \cos\left(\frac{bx^2}{2}\right) dx + i \int_0^{\infty} \sin\left(\frac{bx^2}{2}\right) dx \right) = 2\sqrt{\frac{\pi}{|b|}}\left(\mathcal{C}(\infty) + i\mathrm{sgn}\mathcal{S}(\infty)\right)$

$= \sqrt{\frac{2\pi}{|b|}} e^{i\mathrm{sgn}(b)\frac{\pi}{4}}$

$I_n = \frac{(n-1)!!}{(ib)^{\frac{n}{2}}}\sqrt{\frac{2\pi}{|b|}} e^{i\mathrm{sgn}(b)\frac{\pi}{4}}$

Fresnel integral

cornia spiral, S vs C plot.

Asymptotic estimation of integrals (stationary phase and saddle pt. methods.)

$I_k \int_{-\infty}^{\infty} A(x) e^{ik\phi(x)} dx$

k 'large'

envelope function A, k large $\Rightarrow$ oscillates a tons except when $\frac{d\phi}{dx} = 0$. massive cancelation except near the "stationary points" x_j, where $\phi'(x_j) = 0$

for one stationary point, $x = x_0 + \tau, dx = d\tau$

$\int_{-\infty}^{\infty} A(x_0+\tau) e^{ik\phi(x_0+\tau)} d\tau$

expand A and ϕ in taylor series.

$I_k = \int_{-\infty}^{\infty} [A(x_0) + A'(x_0)\tau + A''(x_0)\frac{\tau^2}{2} + ...] e^{ik[\phi(x_0) + \overbrace{\phi'(x_0)\tau}^{\text{chosen}=0} + \phi''(x)\frac{\tau^2}{2} + ...]} d\tau$

It turns out that leading terms result from using truncated up to 0^{th} order of A and 2^{nd} order of ϕ.

$I_k = A(x_0) e^{ik\phi(x_0)} \int_{-\infty}^{\infty} e^{ik\phi''(x)\frac{\tau^2}{2}} d\tau$

$= A(x_0) e^{ik\phi(x_0)} \sqrt{\frac{2\pi}{k|\phi''(x_0)|}} e^{i\frac{\pi}{4}\mathrm{sgn}(\phi''(x_0))}$

if more tan 1 stationary points, just do summation

k large such that there are many oscillation b/w stationary points.
method of stationary phase.
$\phi(x_i) - \phi(x_j) \gg 2\pi$

---

how to get next order.
expand $e^{ik\left(\phi'''(x_0)\frac{\tau^3}{6}\right)} = 1 + ik\left(\phi'''\frac{\tau^3}{3!}\right) - k^2\left(\phi'''\frac{\tau^3}{3!}\right)^2$
$= e^{ik\phi(x_0)}\int_{-\infty}^{\infty}(\alpha + \alpha\tau + \alpha\tau^2 + ...)e^{ik\left(\phi''(x_0)\frac{\tau^2}{2}\right)}$
4 contribution of order $k^{-\frac{3}{2}}$

$I_k(p) = \int_{-\infty}^{\infty} e^{ik\left(\frac{x^3}{3} - P_x\right)}dx, \qquad P \in \mathbb{R}$
stat. pts. at $x\pm = \pm\sqrt{p}$ only for $p > 0$
$\phi'' = 2x, \phi''(x_\pm) = \pm 2\sqrt{p}$
$I_k(p) = \sqrt{\frac{2\pi}{k \cdot 2\sqrt{p}}}\left(e^{ik\left(\frac{p^{\frac{3}{2}}}{3} - p^{\frac{3}{2}}\right) + i\frac{\pi}{4}} + e^{ik\left(\frac{p^{\frac{3}{2}}}{3} - p^{\frac{3}{2}}\right) - i\frac{\pi}{4}}\right)$
$= \sqrt{\frac{\pi}{k\sqrt{p}}} \cdot 2 \cdot \cos\left(\frac{\pi}{4} - \frac{2}{3}kp^{\frac{3}{2}}\right)$
$A_i(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{\tau^3}{3} + y\tau\right)}d\tau$
$\tau = kx^3, \tau = k^3 x, dx = k^{\frac{1}{3}}d\tau$
$I_k = k^{-\frac{1}{3}}\int_{-\infty}^{\infty} e^{\left(\frac{\tau^3}{3} - \frac{k\tau}{k^{\frac{1}{3}}}p\right)}d\tau = \frac{2\pi}{k^{\frac{1}{3}}}A_i\left(-k^{\frac{2}{3}}p\right)$

---

contribution of second type - edge effects
$I(k) = \int_a^b A(x)e^{ik\phi(x)}dx$
assumes no stationary points b/w a,b
$e^{ik\phi(x)} = \frac{1}{ik\phi'(x)}\frac{de^{ik\phi(x)}}{dx}$
$I(k) = \frac{1}{ik}\int_a^b \frac{A}{\phi'(x)}\frac{d}{dx}e^{ik\phi(x)}dx$
integration by part, $dv = \frac{d}{dx}e^{ik\phi(x)}dx$
$I(k) = \frac{1}{ik}\left[\frac{A(x)}{\phi'(x)}e^{ik\phi(x)}\right]_a^b - \frac{1}{ik}\int_a^b\left(\frac{A}{\phi'}\right)e^{ik\phi}dx$
$I(k) = \frac{1}{ik}\left[\frac{A(x)}{\phi'(x)}e^{ik\phi(x)}\right]_a^b + \mathcal{O}(k^{-2})$
condition - no stationary point b/w a,b, or stationary points far enough away frmo edge.