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OPT 441 (Geometrical Optics)

$i_a = \frac{y_{E_{max}}}{r}$ $y_{E_{max}}$ is max height of ray at center of curvature?

$i_b = -\frac{u_b z}{r}$ z is distance from center of curvature 2 where b-ray cross axis.

$n_{vac} = \frac{c_{vac}}{v}$

n is a function of λ

relative index: $n_{rel} = \frac{c_{air}}{v}$

$t_{AB} = \int_A^B \frac{ds}{v} = \frac{1}{C}\int_A^B n ds$

OPL Optical path length.

Geometrical wavefront - surface where the optical path length is some constant.

Snell's law derived by fermat's principle. (shortest time)

Fix point $a, c.$ vary b

color: blue, yellow, yellow, red

element: hydrogen, helium, sodium , hydrogen

name: F, d, D, C

$\lambda(nm) : 486.1, 587.6, 589.3, 656.3$

$$
\begin{aligned}
S_1 &= -\sum A^2 y_a \Delta\left(\frac{u_a}{n}\right) \\
S_2 &= -\sum AB y_a \Delta\left(\frac{u_a}{n}\right) \\
S_3 &= -\sum B^2 y_a \Delta\left(\frac{u_a}{n}\right) \\
S_4 &= -\sum H^2 c\Delta\left(n^{-1}\right) \\
S_5 &= -\sum\left(\frac{B}{A}\right)\left(B^2 y_a \Delta\left(\frac{u_a}{n}\right) + H^2 c\Delta\left(n^{-1}\right)\right) \\
&= \sum\left[y_b B(2y_a B - y_b A)c\Delta\left(\frac{1}{n}\right) - B^3 y_a \Delta\left(\frac{1}{n^2}\right)\right] \\
S_6 &= -\sum\left(\frac{B^4}{A^2} y_a \Delta\left(\frac{u_a}{n}\right) + 2\left(\frac{B}{A}\right)^2 H^2 c\Delta\left(n^{-1}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{defocus} \quad & W_{020} = \frac{-\epsilon_z}{8n' FNO^2} \\
\text{SA} \quad & W_{040} = \frac{1}{8}S_1 \\
\text{coma} \quad & W_{131} = \frac{1}{2}S_2 \\
\text{astiagmatism} \quad & W_{222} = \frac{1}{2}S_3 \\
\text{field curvature} \quad & W_{220} = \frac{1}{4}(S_3 + S_4) = W_p + \frac{1}{2}W_{222} \\
\text{distortion} \quad & W_{311} = \frac{1}{2}S_5 \\
& W_{440} = \frac{1}{8}S_6
\end{aligned}
$$

$$
\text{Petzval} \quad W_p = \frac{1}{4}S_4
$$

$$\epsilon_y(h,\rho,\phi) = \sigma_1\rho^3\cos\phi + \sigma_2 h\rho^2(2+\cos 2\phi) + (3\sigma_3+\sigma_4)h^2\rho\cos\phi + \sigma_5 h^3 + \frac{2W_{020}\rho_y}{n'u'_a}$$

odd cuz rotation sym $$\epsilon_x(h,\rho,\phi) = \sigma_1\rho^3\sin\phi + \sigma_2 h\rho^2\sin 2\phi + (\sigma_3+\sigma_4)h^2\rho\sin\phi + \frac{2W_{020}\rho_x}{n'u'_a}$$

$$\sigma_n = \frac{S_n}{2n'u'_a} = S_n FNO$$

$$\epsilon_z = \frac{-1}{\rho_y u'_a}\epsilon_y$$

Defocus $$\epsilon_z = \frac{-2W_{020}}{n'u'^2_a} = -8n'W_{020}FNO^2$$

	h	ρ	x-y symmetric	ϵ_y, ϵ_x
Defocus	0	1	Y	
SA	0	3	Y	
Coma	1	2	N	
AST	2	1	N	
PTZ	2	1	Y	
DIST	3	0	N	

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aberration orders
linear: distortion
quadratic: astimatism, field curvature
cubic: coma

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path differential theorem
$dOPL(x,y,z,x',y',z') = n'(k'dx' + L'dy' + M'dz') - n(Kdx + ldy + Mdz)$
perfect point image: all image forming rays meet at a single point (or wavefronts are spherical)
wavefront : $\mathbf{s}' \cdot d\mathbf{r}' = 0$

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ideal image
1 All rays from an object point converge 2 an image point. ("stigmatic" imaging)
object and image points are said to be conjugate.
2 images of all points in a plane $\perp$ to the lens axis lie on a plane that is also $\perp$ 2 the lens axis.
3 Transverse magnification is constant.

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light travels L to R.
+z to the right
ray in I: $u > 0$
ray in II: $u < 0$.
All distances are measured from a reference surface, such as a wavefront or a refracting surface. Distances to the left of the surface are negative.
The refractive power of a surface that makes light rays more convergent is positive. The focal length of such a surface is positive.
The distance of a real object is negative.
The distance of a real image is positive.
Heights above the optic axis are positive.
Angles measured clockwise from the optic axis are negative.

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paraxial form of snell's law:
$nu' = nu - yc\Delta n = nu - y\phi$
object at infinite $\rightarrow u = 0$

$f' = \dfrac{n'}{c\Delta n}$

Suppose image at infinite $u' = 0 \; to f = \dfrac{n}{c\Delta n}$

Def. optical power of a surface as $\phi \equiv c\Delta n = \dfrac{n}{f} = \dfrac{n'}{f'}$ (unit length^{-1},c is curvature)

$\phi_{\text{thin len}} = \phi_1 + \phi_2$

$m = \dfrac{l'}{l}$

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Paraxial transfer eq. $y' = y + \dfrac{t}{n} nu$

General: $n'_j u'_j = n_j u_j - y_j \phi_j$

$y_{j+1} = y'_j + \dfrac{t'_j}{n'_j} n'_j u'_j$

$\phi = \dfrac{n'}{l'} - \dfrac{n}{l}$ (lens eq. $\dfrac{1}{f} = \dfrac{1}{o} + \dfrac{1}{i}$)

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of axis points

refract: $n'u' = -\dfrac{n(y - y_0)}{l} - y\phi$

$= \dfrac{ny_0}{l} - \dfrac{n'y}{l'}$

$y_i = \left(\dfrac{n}{n'}\right)\left(\dfrac{l'}{l}\right) y_0$

$m = \dfrac{n}{n'} \dfrac{l'}{l}$

(linear mapping)as objects get smaller, and aperture get smaller → paraxial imagery.

paraxial eq. generate an ideal imae

Front principal plane, rear principal plane P'. P, P' probably between the whole optical system.

rear focal length f' is the distance from P' to F' (can be ±).

Back focal distance (BFD) is the distance from V' (end of optical system) to F'.

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location of principal planes

$\dfrac{P'V'}{n_3} = \left(\dfrac{t_2}{n_2}\right) \dfrac{\phi_1}{\phi}$

$\dfrac{PV}{n_1} = -\left(\dfrac{t_2}{n_2}\right) \dfrac{\phi_2}{\phi}$

ϕ_1, ϕ_2 surface powers, ϕ total thick lens power.

easier to trace paraxial ray (matrix) and find intersection.

if want to find focal length: $f'_{sys} = -\dfrac{y_1}{u'_{sys}}$

Newton's form of imaging relationship $ff' = zz'$

For magnification $m = 1$, ray striking front principal plane (at any angle and any height), will leave rear principal plane at the same height.

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nodal points: ray in object space through the front nodal point leaves the lens at the rear nodal point at the same angle. (points of unity angular magnification)

$\overline{PN} = f' - f$

if $f' = f$ or equivalently $n' = n$, then N is at P.

Matrix

$P'_j = \begin{pmatrix} y'_j \\ n'_j u'_j \end{pmatrix}$

prime indicates on image side of surface.

Translation matrix $T, det(T)=1$

$T_j = \begin{pmatrix} 1 & \frac{t'_j}{n'_j} \\ 0 & 1 \end{pmatrix}$

Then $P_{j+1} = T_j P'_j \qquad \begin{pmatrix} y_{j+1} \\ n_{j+1}u_{j+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{t'_j}{n'_j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y'_j \\ n'_j u'_j \end{pmatrix}$

Refraction matrix $R, P'_j = R_j P_j, det(R) = 1$

$R_j = \begin{pmatrix} 1 & 0 \\ -\phi_j & 1 \end{pmatrix}$

$P' = T_k R_k ... R_2 T_1 R_1 T_0 P = MP$ (If we know 3 entries of M, we can solve for 4^{th} using det)

$$M = \begin{pmatrix} 1 - \frac{t}{n}\phi_1 & \frac{t}{n} \\ \frac{t}{n}\phi_1\phi_2 - \phi_1 - \phi_2 & 1 - \frac{t}{n}\phi_2 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{t_2}{n_2}\phi & \frac{t_1}{n_1} + \frac{t_2}{n_2} - \frac{t_1 t_2}{n_1 n_2}\phi \\ -\phi & 1 - \frac{t_1}{n_1}\phi \end{pmatrix}$$

ABCD rule (also holds for Gaussian laser beams, radius of curvature treated as complex quantity though):$-\frac{R'}{n'} = \frac{A(-\frac{R}{n}) + B}{C(-\frac{R}{n}) + D}$

The Gaussian constant (depending on how ppl define it) can mean $R_k ... R_1$

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plane shift by $+t$ on the object space, plane $+t'$ in the image plane.

$$\begin{pmatrix} \hat{y}' \\ n'u' \end{pmatrix} = \begin{pmatrix} 1 & \frac{t'}{n'} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -\frac{t}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{y} \\ nu \end{pmatrix}$$

$$= \begin{pmatrix} \hat{y}' \\ n'u' \end{pmatrix} = \begin{pmatrix} A + \frac{t'}{n'}C & B - \frac{t}{n}A + \frac{'}{n'}D - \frac{tt'}{nn'}C \\ C & D - \frac{t}{n}C \end{pmatrix} \begin{pmatrix} \hat{y} \\ nu \end{pmatrix}$$

since the determinant = 1, M^{-1}: swap A_{11}, A_{22}. swap + negate A_{12}, A_{21}.

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rear focal point: ray with $nu = 0$ cross axis in image space. $\hat{y}' = \left(A + \frac{t'}{n'}C\right)\hat{y}$

Solve for t' such that $\hat{y}' = 0$: $z' = -\frac{n'A}{C}$

Front focal point: set $n'u' = 0$, solve for t such that $\hat{y} = 0$

$z = \frac{nD}{C}$

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principal plane

in general, the planes $z = t$, $z' = t'$ are conjugate when $B - \frac{t}{n}A + \frac{t'}{n'}D - \frac{tt'}{nn'}C = 0$

magnification m : $m = A + \frac{t'}{n'}C$

principal planes are conjugate planes with

$$m = 1 \qquad \Rightarrow \qquad t' = \frac{n'(1-A)}{C} \tag{1}$$

to find t:

$$B + \frac{t'}{n'}D - \frac{t}{n}\underbrace{\left(A + \frac{t'}{n'}C\right)}_{=m=1} = 0 = B + \frac{t'}{n'}D - \frac{t}{n}$$

$$\text{plug in eq (??)} \qquad \frac{t}{n} = B + \frac{D}{n'} \cdot \frac{n'(1-A)}{C}$$
$$= B + \frac{D(1-A)}{C}$$

$$= \frac{BC + D - AD}{C}$$
$$= \frac{D - (AD - BC)}{C}$$
$$= \frac{D-1}{C}$$
$$\therefore t = \frac{n(D-1)}{C}$$

plane $z = \frac{n(D-1)}{C}$ is the front principal plane.
plane $z' = \frac{n'(1-A)}{C}$ is the rear principal plane.
front focal length $f = \frac{n(D-1)}{C} - \frac{nD}{C} = -\frac{n}{C}$
rear focal length $f' = -\frac{n'(1-A)}{C} - \frac{n'A}{C} = -\frac{n'}{C}$
$-C = \frac{n}{f} = \frac{n'}{f'} \qquad \Rightarrow \qquad c = -\phi_{sys}$

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start with $\hat{y} = 0$, find tt' such that $\hat{y}'$ vanishes and $u = u'$

$$\hat{y} = 0 \qquad \Rightarrow \qquad \hat{y}' = \left(B - \frac{t}{n}A + \frac{t'}{n'}\left(D - \frac{t}{n}C\right)\right) nu$$
$$n'u' = \left(D - \frac{t}{n}C\right) nu$$

set $\hat{y'} = 0, u = u' \qquad \Rightarrow \qquad t = \frac{nD - n'}{C};\ t' = \frac{n - n'A}{C}$
$z = \frac{nD - n'}{C}$ is the front nodal point
$z' = \frac{n - n'A}{C}$ is the rear nodal point
note that when $n = n'$, nodal points and principal points coincide.

Given the matrix the optical systems (Matrix M) b/w planes $z = 0, z' = 0$

important points	values
F	$\frac{nD}{C}$
F'	$-\frac{n'A}{C}$
P	$\frac{n(D-1)}{C}$
P'	$\frac{n'(1-A)}{C}$
N	$\frac{nD-n'}{C}$
N'	$\frac{n-n'A}{C}$
f	$-\frac{n}{C}$
f'	$-\frac{n'}{C}$

$\star f$ measured from P, f' measured from P' (sign can change if there's mirror). F, P, N measured from $z = 0$ (start of system), F', P', N' measured from $z' = 0$ (end of system). $(P = F + f, P' = F' - f')$(Notice when $n = n'$, $P = N$, $P' = N'$.)
For matrix M, $B = 0 \qquad \Rightarrow \qquad z = 0, z' = 0$ are conjugate and $m = A$.
$A = 0 \qquad \Rightarrow \qquad z' = 0$ is at F', $f = nB, f' = n'B$

$D=0 \quad \Rightarrow \quad z=0$ is at F and $f=nB, f'=n'B$
$C=0 \quad \Rightarrow \quad$ system is afocal. image of an infinitely distant object is also at infinity.

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word about thin lens
$M=\lim_{t\to 0}\begin{pmatrix} 1 & 0 \\ -c_2(1-n) & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{t}{n} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c_1(n-1) & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -(c_1-c_2(n-1) & 1 \end{pmatrix}$
$\phi=(c_1-c_2)(n-1)$
all thin lenses with same value of (c_1-c_2) are equivalent.

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mirror system
Refraction matrix $R=\begin{pmatrix} 1 & 0 \\ -c\Delta n & 1 \end{pmatrix}$
For reflection $R_{reflect}=\begin{pmatrix} 1 & 0 \\ 2nc & 1 \end{pmatrix}$

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mirror example (in air)
ray (from left) hit mirror_1, mirror_2. mirror_1 on the right of mirror_2
t'_1 is negative
After 2 reflection $\begin{pmatrix} y' \\ n'u' \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -2c_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -t'_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2c_1 & 1 \end{pmatrix}\begin{pmatrix} y \\ nu \end{pmatrix}$

Entrance pupil: image of a stop that is formed in object space by the lenses that precede the stop (SA of entrance pupil is based on the ray in (virtual) object space, the actual ray be larger than SA).
Exit pupil: image of a stop that is formed in image space by the lenses following the stop (SA based on ray in (virtual) image space).
aperture stop , entrance pupil, exit pupil are all conjugate.
when aperture stop lies physically in object space $\Rightarrow$ stop corresponds with entrance pupil.

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lens ($n=n'=1$):

$$\begin{pmatrix} y' \\ u' \end{pmatrix}=\begin{pmatrix} 1 & t'_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}\begin{pmatrix} 1 & t'_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$$
$$=\begin{pmatrix} 1-t'_1\phi & t'_0+t'_1-t'_0t'_1\phi \\ -\phi & 1-t'_0\phi \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$$

conjugate occur when $B=0 \Rightarrow m=A$, $t'_0=\dfrac{-t'_1}{1-t'_1\phi}$
$t'_0<0 \Rightarrow$ entrance pupil lies to the right of the lens $\Rightarrow$ virtual entrance pupil.
1) stop at lens - entrance and exit pupils also at lens
2) stop b/w lens and F', virtual entrance pupil
3) stop at F', entrance pupil at infinity
4) stop following F', real entrance pupil
telecentric in object space - entrance pupil at infinity.
telecentric in image space - exit pupil at infinity.
Doubly telecentric - both at infinity.

Given matrix M of a system. To see its effect on all kinds of ray: we take linear independent ray $\vec{A},\vec{B}$. The effect of any other rays is $c_0\vec{A}+c_1\vec{B}=M(c_0\vec{A}+c_1\vec{B})$

Marginal ray (axial ray, a ray) - ray from axial object point passing through the edge of the aperture stop
Chief ray (principal ray, b ray) - a ray from the edge of the field (Edge of the object) passing through the center of the aperture stop.

$SA=|y_a|+|y_b|$

$y_{+2} = y_{a2} + y_{b2} \qquad y_{-2} = -y_{a2} + y_{b2}$
beam diameter $= y_{+2} + y_{-2} = 2|y_{a2}|$

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vignetting factor - how much of the diameter of the beam is outside the lens aperture, as a fraction of the beam diameter.
$V^{\pm} = \frac{1}{2} - \frac{SA - |y_b|}{2|y_a|}$
If $y_b > 0, V^- = 0$, use V^+ only. If $y_b < 0, V^+ = 0$, use V^- only.
$V^{\pm} = max(V_j^{\pm})$
V factor for system is $V_F = V^+ + V^-$.

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speed of system (pg. 53)
F-number and numerical aperture
$FNO_{eff} = \left| \frac{1}{2n'u'_a} \right|$
for object at ∞: $FNO_{eff} \equiv \frac{f'}{n'CA}$
CA: entrance pupil clear aperture.

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numerical aperture (NA)
$NA \equiv |n' \sin U'_a|$ (in image space)
$n \sin U_a$ (in object space)
real ray angle U is the angle
paraxial angle u is $\tan\theta$ (only u make paraxial eq. correct)

Paraxial invariant, Lagrange Invariant
for 2 rays $(y, nu), (\overline{y}, n\overline{u})$, the quantity $I = n\overline{u}y - nu\overline{y}$ is invariant
Lagrange invariant $H \equiv nu_b y_a - nu_a y_b$
at object,image $H = -nu_a y_b$
at pupils, stop $H = nu_b y_a$
at object, $H = -nu_a y_b$
At image $H = -n'u'_A y'_b$
Thus $m = \frac{y'_b}{y_b} = \frac{nu_a}{n'u'_a}$
$H^2 = \underbrace{n'^2 u'^2_a}_{E \propto NA^2} \underbrace{y'^2_b}_{\propto \text{Area}}$
Diffraction theory - image of a point is given by the Airy pattern (Assuming circular pupil)
distance from peak to the first zero is the radius of "airy disk" $\frac{-0.61\lambda}{n'u'_a}$ $\quad u'_a$ usually negative.
Rayleigh criterion for resolution - 2 points "barely resolved" when peak of 1 image falls on the first zero of the other: 2 points separated by $\frac{0.61\lambda}{|n'u'_a|}$
It is of the form $-\frac{R\lambda}{n'u'_a}$
number of spots across an image diameter is $N_{spots} = \frac{2H}{R\lambda}$

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afocal - system with both object, image at infinity.
$m_{afocal} \equiv \frac{n'u'_b}{nu_b} = \frac{y_a}{y'_a}$

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with magnifier: $u'_b = \frac{-y_{b,object}}{f}$
without magnifier: $u_b = \frac{-y_{b,object}}{250mm}$

$m_{visual} = \frac{250mm}{-y_{b,object}} = \frac{n'}{n} \cdot \frac{250mm}{f} = \frac{250mm}{f}$

"eye relief" - distance from last surface of system 2 entrance pupil of eye.
eye can focus b/w 250mm to ∞ entrance pupil of eye $2 - 8mm$
exit pupil of system and entrance pupil of eye should coincide and approx the same size. (if not, vignetting restricts field of view, wastes light.)
lens system typically is 10-20 mm away from eye. (comfortable distance - "eye relief") (should be larger fror some system - rifle scopes)
angular resolution of the eye is $\sim$ 1minute of arc $\approx$ 0.3milliradians

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Galilean telescope
for 2 thin lense in air, ϕ_1, ϕ_2, separated by distance t
system power is $\phi_{sys} = \phi_1 + \phi_2 - t\phi_1\phi_2$
afocal system: $\phi_{sys} = 0$
$t = f_1 + f_2$
$m_{afocal} = -\frac{f_1}{f_2} = -\frac{f_o}{f_e}$
positivev since $f_2 < 0$
system has an internal exit pupil.
vignetting occurs almost immediately as object moves off axis, and system ha ssmall usable field of view.

Keplerian telescope
$f_1 > 0, f_2 > 0 \Rightarrow m_{afocal} < 0$
inverted imageexternal exit pupil, eye can be placed in pupil
field of view limited by vignetting caused by eyepiece or by field stop paced at the internal image
erecting prism 2 re–invert image.

Newtonian telescope: parabolic mirror (perfect image for parallel rays along axis)
Cassegrain telescope is the reflecting equivalent of a telephoto lens. positive primary, negative secondary. (parabolic primary, hyperbolic secondary)
Gregorian telescope - parabolic primary mirror of positive power, ellipsidal (positive) secondary mirror.

$|m| = \left|\frac{-z'}{f}\right| = \frac{f_{tlens}}{f_o}$
z' tube length
system magnifying power is $\frac{\text{tubelength} \cdot 250mm}{f_o f_e}$
infinity corrected: image at ∞, objective followed by tube lens followed by eyepiece.
$-\frac{f_{tlens}}{f_o}$
image projected onto detector plane is independent of object position. (if telecentric)

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direction cosines
$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_Z = 1$
define K, L, M to be the cos.
for point P_{-1} on plane $z = z_{-1}$ corresponds to the point P on $z = z_0$
$x_0 = x_{-1}, y_0 = y_{-1}, z_0 = -t'_{-1} + z_{-1}$
$(x_0, y_0, z_0) + \Delta < K, L, M >$
intersect $z = \frac{1}{2}c(x^2 + y^2 + z^2)$
$c\Delta^2 - 2\Delta[M - c(Kx_0 + Ly_0 + Mz_0)] + c(x_0^2 + y_0^2 + z_0^2) - 2z_0 = 0$
Let $G = \Delta[M - c(Kx_0 + Ly_0 + Mz_0)], E = c(x_0^2 + y_0^2 + z_0^2) - 2z_0$
solution of quad is then $c\Delta = G \pm \sqrt{G^2 - cE}$

Take the Δ_- solution.
multiply by conjugate over conjugate: then we have

$$\Delta = \frac{E}{G + \sqrt{G^2 - cE}}$$

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snell's law 3D
derivation:
path P, P' passes through P_S that satisfy $z_s = f(x_s, y_s)$
use fermat's principle: $OPL = \int_P^{P'} nds = n\sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2} + n'\sqrt{(x'-x_s)^2 + (y'-y_s)^2 + (z'-z_s)^2}$
Let $d \equiv \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2}, d' \equiv \sqrt{(x'-x_s)^2 + (y'-y_s)^2 + (z'-z_s)^2}$
Fermat's requires $\frac{\partial}{\partial x_s}(nd + n'd') = \frac{\partial}{\partial y_s}(nd + n'd') = 0$
$\frac{n(x_s - x)}{d} + \frac{n(z_s - z)}{d} \left.\frac{\partial f}{\partial x}\right|_{x_s, y_s} + \frac{n'(x_s - x')}{d'} + \frac{n'(z_s - z')}{d'} \left.\frac{\partial f}{\partial x}\right|_{x_s, y_s} = 0$
eq. apply to y too.
direction cosines: $K = \frac{x_s - x}{d}, L = \frac{y_s - y}{d}, M = \frac{z_s - z}{d}$
$K' = \frac{x' - x_s}{d'}, L = \frac{y' - y_s}{d'}, M = \frac{z' - z_s}{d'}$
rewrite as $(nK - n'K') + (nM - n'M') \left.\frac{\partial f}{\partial x}\right|_{x_s, y_s} = (nL - n'L') + (nM - n'M') \left.\frac{\partial f}{\partial y}\right|_{x_s, y_s} = 0$
Then $\frac{\partial f}{\partial y}(1) - \frac{\partial f}{\partial x}(2)$ yields
$(nK - n'K') \left.\frac{\partial f}{\partial y}\right|_{x_s, y_s} - (nL - n'L') \left.\frac{\partial f}{\partial x}\right|_{x_s, y_s} = 0$
normal vector to surface: $\mathbf{N} \equiv \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$
$(n\mathbf{r} - n'\mathbf{r}) \times \mathbf{N} = [0\ 0\ 0]^T$
Law of refraction is then $(n\mathbf{r} \times \mathbf{N} = n'\mathbf{r} \times \mathbf{N}$
$n \sin I = n' \sin I'$
$\mathbf{N} = (\alpha, \beta, \gamma), |\mathbf{N}| = 1$
$n'K' = nK + \Gamma\alpha \qquad (L, \beta), n'M' = nM - c\Gamma z + \Gamma$
$\Gamma = n' \cos I' - n \cos I$
$F(x, y, z) = z - \frac{1}{2}c(x^2 + y^2 + z^2)$
$\mathbf{N} = (-cx, -cy, 1 - cz)$
$\cos I = \mathbf{r} \cdot \mathbf{N} = \sqrt{G^2 - cE}$
$n' \cos I' = \sqrt{n'^2 - n^2(1 - \cos^2 I)}$

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dispersion
color: blue, yellow, yellow, red
element: hydrogen, helium, sodium , hydrogen
name: F, d, D, C
$\lambda(nm) : 486.1, 587.6, 589.3, 656.3$
dispersion relevant to optical design: $\frac{\phi_F - \phi_C}{\phi_d} = \frac{\Delta\phi}{\phi}$
'reciprocal dispersive power' $= \frac{\phi_d}{\phi_F - \phi_C} = \nu_d = \frac{n_d - 1}{n_F - n_C}$

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abbe ν number
$\nu_d == \frac{\phi_d}{\phi_F - \phi_C} = \frac{n_d - 1}{n_F - n_C}$

517646 - code for abbe prism
first 3 number describe the index, last 3 describe dispersion. .
dispersion 64.2

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longitudinal chromatic aberration (thin lenses in air)
$\Delta\phi = \phi_F - \phi_c = \left(\frac{l'_c - l'_F}{l'_F l'_c}\right) - \left(\frac{l_c - l_F}{l_F l_c}\right)$
for small aberrations $l_c l_F \approx l_d^2, \qquad l'_c l'_F \approx l'^2_d$
$\Rightarrow \Delta\phi = \frac{\phi}{\nu} = \frac{l'_c - l'_F}{l'^2} - \frac{l_c - l_F}{l^2}$
(no subscript $\Rightarrow$ d line)
define $\Delta l \equiv l_F - l_c, \Delta l' = l'_F - l'_c$
multiply by $-y_a^2$, since $u_a^2 = \frac{y_a^2}{l^2}$, $u'^2_a = \frac{y'^2_A}{l'^2}$
then $-\frac{y_a^2 \phi}{\nu} = \Delta l'(u'^2_a) - \Delta l(u_a^2)$
happen at each lenses: summation.
$\Delta l' = \Delta l$ for nex lens, all terms cancel but first, and last
$\sum_{lenses}\left(\frac{-y_{aj}^2\phi_j}{\nu_f}\right) = \underbrace{L'_{CH}u'^2_a}_{\text{image space}} - \underbrace{L_{CH}u_a^2}_{\text{object}}$

normally, one has unaberrated object, so $L_{CH} = 0$, then wehave $L'_{CH} = -\frac{1}{u'^2_a}\sum_{lenses}\frac{y_{aj}^2\phi_j}{\nu_j}$

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thin achromat:

$$\phi_1 = \frac{\phi_{sys}\nu_1}{\nu_1 - \nu_2}$$
$$\phi_2 = \frac{\phi_{sys}\nu_2}{\nu_2 - \nu_1} = \frac{-\phi_1\nu_2}{\nu_1}$$

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transverse chromatic Aberration for thin lens in air
$y'_{b,F} - y'_{b,C} = T'_{CH} = \frac{1}{u'_a}\frac{y_a y_b \phi}{\nu}$
$T'_{CH} = \frac{1}{u'_a}\sum_{lenses}\frac{y_{aj}y_{bj}\phi_j}{\nu_j}$
$T'_{CH} = -u'_{a,sys}\left(\frac{y_{bj}}{y_{aj}}\right)L'_{CHj}$

No primary axial color if $\sum\frac{y_a^2\phi}{\nu} = 0$
no primary lateral color if $\sum\frac{y_a y_b\phi}{\nu} = 0$
no secondary axial color if $\sum\frac{y_a^2 P\phi}{\nu} = 0$
no secondary lateral color if $\sum\frac{y_a y_b P\phi}{\nu} = 0$

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secondary spectrum
$P_{x,y} \equiv \frac{n_x - n_y}{n_F - n_C}$
$P_{d,F} = \frac{n_d - n_F}{n_F - n_c}$

Monochromatic aberrations (misnomer)

$h \equiv= \dfrac{y \text{ object position}}{\text{max object height}}$
$p \equiv \dfrac{\text{radial pupil position}}{\text{pupil radius}}$
$0 \leq h, \rho \leq 1$
aberration are functions of $h, \rho, \cos\phi$
any particular transverse error $\vec{\epsilon} = (\epsilon_x, \epsilon_y)$
Define wavefront aberration function to bee the optical path difference (OPD) between the actual wavefront and the reference sphere, then the physical separation between the two surfaces in image space is geven by $\dfrac{W(h, \rho, \phi)}{n'}$. measured along the aberrated ray.
W (+) $\Rightarrow$ actual wavefront leads the reference sphere.
E' : center of exit pupil has ideal chief ray $\overline{E'P'_0} \equiv R$.
The coordinates are $Q_0(x, y, z), P'(\epsilon_x, y'_0 + \epsilon_y), P'_0(0, y'_0)$.
direction cosine of ray $Q_0 \to Q \to P'$.
final result: $\epsilon_x = \dfrac{1}{n'u'_a}\dfrac{\partial W}{\partial \rho_x}, \epsilon_y = \dfrac{1}{n'y'_a}\dfrac{\partial W}{\partial \rho_y}$

W must be rotational invariant
general form of W is $W(h, \rho, \cos\phi) = \sum W_{ijk} h^i \rho^j \cos^k \phi$
symmetry condition: $i + j$ must be even.
$n = i + j$ define the net power of the length component, is called the order of the aberration.
transverse ray aberrations are $\propto \dfrac{\partial W}{\partial p}$. So ray aberrations are one order lower - odd.
3^{rd} order - Seidel aberrations. or primary aberration. 5^{th} as secondary. 7^{th} as tertiary...
this class only concerned with order up to 4:
$W(h, \rho, \cos\phi) = W_{000} + W_{020}\rho^2 + W_{111}h\rho\cos\phi + W_{200}h^2 + W_{040}\rho^4 + W_{131}h\rho^3\cos\phi + W_{220}h^2\rho^2 + W_{222}h^2\rho^2\cos^2\phi + W_{311}h^3\rho\cos\phi + W_{400}h^4$
W_{ijk} are called wavefront aberration coeff. (unit of wavelength)
(ϵ_x, ϵ_y) usually given in length unit.
transverse ray aberration dont depend on the h^2 terms.

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piston error (is usually ignored) are $W(h^2) = W_{000} + W_{200}h^2 + ...$
at any object point, these terms only add a constant path error across the pupil.

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defocus
$\epsilon_y = \dfrac{2W_{020}\rho_y}{n'u'_a}, \epsilon_z = \dfrac{-1}{\rho_y u'_a}\epsilon_y$
looking at only the $W_{020}\rho^2$ term, we have $\epsilon_z = \dfrac{-2W_{020}}{n'u'^2_a}$
$\epsilon_z = -8n'W_{020}FNO^2$
$W_{020} = \dfrac{-\epsilon_z}{8n'FNO^2}$

Rayleigh's criterion for diffraction limited
$|W_{max}| \leq \dfrac{\lambda}{4}$.
applying to defocus, $W_{max} = W_{020}$
$\epsilon_z = \pm 2\lambda_0 FNO^2$
$|\epsilon_z| \leq |2\lambda_0 FNO^2|$

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aberration analysis
trace ray fans (Y-fan, X-fan).
plot $\epsilon_y vs. \rho_y, \epsilon_x vs. \rho_x$.

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a planar object focus on a curve surface.

Petzval sum - $\frac{1}{n'R_p} = \sum c\Delta\left(\frac{1}{n}\right)$
conventional case of a plane object, we denote the image space value of R' as $R_p$for 'Petzval radius'.
$W = W_p h^2 \rho^2$.
$\epsilon_z = \frac{y'^2}{2R_p} = \frac{-2W_p h^2}{n' u_a'^2} = \frac{1}{2} n' \sum c\Delta\left(\frac{1}{n}\right) y_b'^2 h^2$.
$W_p = -\frac{1}{4} H^2 \sum c\Delta(n^{-1})$

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y is the height of the surface
s is the s displacement of the surface at $y = y$
notice that $s = \frac{y^2}{2r}\mathcal{O}(4)$, the s^4 terms in $W_{4^{th}\text{ order}}$ gives 6^{th} order, so we can ignore s^2.
$W_{4^{th}\text{ order}} = \frac{1}{8}(y^2+s^2)^2 n^2 \left(\left(\frac{1}{n'l'} - \frac{1}{nl}\right)\left(\frac{1}{l} - \frac{1}{r}\right)^2\right) = \frac{1}{8} y^4 n^2 \left(\left(\frac{1}{n'l'} - \frac{1}{nl}\right)\left(\frac{1}{l} - \frac{1}{r}\right)^2\right)$
$W_{4^{th}\text{ order}} = 0$ if
1) $l = r = l'$ (object centered surface
2) $y = 0$ object at the surface (like field lens)
3) $nl = n'l'$ (aplanatic point)
–
$W_{6^{th}\text{ order}} = -\frac{1}{16}(y^2+s^2)^3 n^3 \left(\left(\frac{1}{n'^2 l'^2} - \frac{1}{n^2 l^2}\right)\left(\frac{1}{l} - \frac{1}{r}\right)^3\right)$
$y^4 = y_T^4 + \mathcal{O}(6) = y_T^4 = (\rho y_a)^4$
$W_{4^{th}\text{ order}} = \frac{1}{8} n^2 y_a \left(\Delta\left(\frac{-u_a}{n}\right)(-u_a + \alpha)^2\right)\rho^4$
angle of incidence of a-ray: $i_a = u_a - \alpha$
Let $s_1 \equiv -\sum_{\text{surfaces}} A^2 y_a \Delta\left(\frac{u_a}{n}\right), A \equiv n i_a$
$\Rightarrow W_{4^{th}\text{ order}} = W_{040}\rho^4 \Rightarrow W_{040} \equiv \frac{1}{8}s_1$
transverse ray aberration:
$\epsilon_y = \sigma_1(\rho_y^3 + \rho_x^2\rho_y), \epsilon_x = \sigma_1(\rho_x^3 + \rho_y^2\rho_x)$
where $\sigma_1 \equiv \frac{4@_{040}}{n'u_a'} = \frac{s_1}{2n'u_a'}$
terminology: $\sigma_1 < 0$: undercorrected spherical aberration
$\sigma_1 > 0$: over corrected spherical aberration.

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page 148 image

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coma
$W_{131} \underbrace{h\rho_y}_{\Delta m} \rho^2$
may be thought of as a magnification error that varies quadraticall with zone in the pupil.
coma $\propto \sum ABy_a\Delta\left(\frac{u_a}{n}\right)$
vanishes if
1) $\Delta\left(\frac{u_a}{n}\right) = 0$ (aplanatic)
$y_a = 0$ (object at surface
3) $B = 0$ (pupil cenered srface
4) $A = 0$ (object-centered surface.
system (and a-ray and b-ray) is symmetric
transverse ray aberration for coma: $\epsilon_y = \sigma_2 h(3\rho_y^2 + \rho_x^2), \epsilon_x = \sigma_2 h(2\rho_x\rho_y)$
where $\sigma_2 \equiv \frac{W_{131}}{n'u_a'} = \frac{s_2}{2n'u_a'}$

$\epsilon_y = \sigma_2 h\rho^2(2+\cos 2\phi), \epsilon_x = \sigma h\rho^2 \sin 2\pi$
center of circle: $2\sigma_2 h\rho^2$ from paraxial image point
$r = \sigma_2 h\rho^2 2$.
Aplanatism is the corresponding stationarity condition when the pencil of rays hass zero aberration at he original object point. Aplanatic - freedom from both spherical aberration and coma.
optical cosine rule: $n'\frac{dr'}{dr}\cos\theta' - n\cos\theta = c$
abbe sine condition
cosine rule for $\theta = \frac{\pi}{2} = \theta'$:
$\frac{n\sin U}{n'\sin U'} = m$
cosine rule for $\cos U = \cos U' = 1$ (both U, U' approaches 0, from (-), (+))
$\Rightarrow m_L = \frac{n}{n'}\left(\frac{\sin\left(\frac{U}{2}\right)}{\sin\left(\frac{U'}{2}\right)}\right)^2$
known as Herschel condition.
$m_L = \frac{n'}{n}m^2 \Rightarrow m = \frac{n\sin\left(\frac{U}{2}\right)}{n'\sin\left(\frac{U'}{2}\right)}$
but incompatible with the abbe sine condition unless $U = U'$, while implies that $m = m_L = \frac{n}{n'}$

$W = W_{222}h^2\rho_y^2$
$\epsilon_x = 0, \epsilon_y = 2\sigma_3 h^2\rho_y, \sigma_3 = \frac{S_3}{2n'u'_a}$

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field curvature
$W = W_{220}h^2\rho^2$

Astigmatism, field curvature, defocus
$W = W_{020}\rho^2 + W_{222}h^2\rho^2\cos^2\phi + W_{220}h^2\rho^2$
$\epsilon_y = (-u'_a\epsilon_z + (3\sigma_3+\sigma_4)h^2)\rho_y$
$\epsilon_x = (-u'_a\epsilon_z + (\sigma_3+\sigma_4)h^2)\rho_x$.
paraxial focus: $\epsilon_z = 0$,
Sagittal focus: $\epsilon_x = 0$. ($\epsilon_z = \frac{(\sigma_3+\sigma_4)h^2}{u'_a}, \epsilon_y = 2\sigma_3 h^2\rho_y$ (length $4\sigma_3 h^2$))
Tangential focus: $\epsilon_y = 0$. ($\epsilon_z = \frac{(3\sigma_3+\sigma_4)h^2}{u'_a}, \epsilon_y = -2\sigma_3 h^2\rho_x$ (length $4\sigma_3 h^2$))
Medial focus: circle: $\epsilon_z = \frac{2(\sigma_3+\sigma_4)h^2}{u'_a}$
$\epsilon_y = \sigma_3 h^2\rho_y, \epsilon_x = -\sigma_3 h^2\rho_x$
$r = 2\sigma_3 h^2$

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astimatism depends on incidence angles of ray as well as the powers and indices.

Distortion
transverse ray error
$\epsilon_y = \frac{s_5}{2n'u'_a}h^3 = \sigma_5 h^3$
$\epsilon_x = 0$
distortion $\frac{\epsilon_y}{y'}$ in percent.
$\frac{\epsilon_y}{hy'_b} = \frac{\sigma_5 h^2}{y'_b}$ in percentage.

S_5 vaishes when
1) $B = 0$,

2) If the system (a,b ray) is symmetric.
distortion usually classified by considering the image of a square grip:
for $\sigma_5 < 0$, we have 'barrel distortion'
(looking like a front square grip of a sphere)
for $\sigma_5 > 0$, we have 'pincushion distortion'
(looking like a back square grip of a sphere)

Eccentricity
E defined as
$HE = \frac{y_b}{y_a}$
$E_{j+1} - E_j = \frac{t'_j}{n'_j y_{a_j} y_{aj+1}}$

Stop-shift effec.
$E = \frac{y_b}{Hy_a}$, $E = 0$ at stop.
shifting the stop by ΔE (Old stop E changes from 0 to $\Delta E = \frac{\Delta y_b}{Hy_a}$), the general result is that ΔE is an invariant quantity throughout the system.
$\Delta S = H\Delta E = \frac{\Delta y_b}{y_a}$
$\frac{B}{A} = H\left(E + \frac{1}{Ay_a}\right)$
changing E by ΔE yields the change: $\frac{B}{A} = H\left(E + \Delta E + \frac{1}{Ay_a}\right)$
The ratio $\frac{B}{A}$ occurs in sevveral relationship:
$s_2 = \frac{B}{A}s_1, s_3 = \left(\frac{B}{A}\right)^2 s_1, s_5 = \frac{B}{A}(s_3 + s_4), T'_{CH} = -u'_a \frac{B}{A} L'_{CH}$

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stop shift eqs: $\Delta S = H\Delta E = \frac{\Delta y_b}{y_a}$
$\delta s_1 = 0$
$\delta s_2 = (H\Delta E)s_1$
$\delta s_3 = 2H\Delta E s_2 + (H\Delta E)^2 s_1$
$\delta s_4 = 0$
$\delta s_5 = H\Delta E(3s_3 + s_4) + 3(H\Delta E)^2 s_2 + (H\Delta E)^3 s_1$
$\delta L_{CH} l = 0$
$\delta T'_{CH} = -u'_a \cdot H\Delta E \cdot L'_{CH}$

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thin lens quantities
shape factor or bending factor β
$\beta = \frac{c_1 + c_2}{c_1 - c_2}$
conjugate factor or magniication ffactor γ
$\gamma = \frac{u_{a1} + u'_{a2}}{u_{a1} - u'_{a2}} = \frac{m+1}{m-1}$
object at infinity $u_{a1} = 0 \Rightarrow \gamma = -1$

---

thin lens aberrations (page 176,177 for exact expressions)
$S_1 \propto \beta^2, \gamma^2$
$S_2 \propto \beta, \gamma$
$S_3 = H^2\phi$

$S_4 = \frac{H^2\phi}{n}$
$S_5 = 0$
page 177
aberrations in terms of β, γ.

Aspheric contributions to the seidel aberrations
putting a coating on a spheric surface to make it aspheric
$a = \kappa c^3 y_a^4 \Delta n$ (conic)
$a = 8 d y_a^4 \Delta n$ (polynomial)
$\delta S_1 = a$
$HE = \frac{y_b}{y_a}$
$\delta S_1 = a$
$\delta S_2 = \frac{y_b}{y_a} a$
$\delta S_3 = \left(\frac{y_b}{y_a}\right)^2 a$
$\delta S_4 = 0$
$\delta S_5 = \left(\frac{y_b}{y_a}\right)^3 a$
no effect on Petzval curvature S_4, or first order chromatic aberrations (L'_{CH}, Y'_{CH})

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if we want 0 aberration on that surface for S_1: $S_1 + \delta S_1 = 0$

$\Delta\left(\frac{u_a}{n}\right) = 0, nl = n'l' \qquad \Rightarrow \qquad S_1 = S_2 = S_3 = 0$
$\Rightarrow m = \frac{nl'}{n'l} = \frac{n^2}{n'^2}$

$A = ni_a = 0 \Rightarrow S_1 = S_2 = 0$
$m = \frac{nl'}{n'l} = \frac{n}{n'}$
For 'aplanatic thin lens',
if $n = n_g$(index of glass)$, n' = 1 \Rightarrow m = m_{\text{apl.srf.}} \cdot m_{\text{obj.cent.srf.}} = \left(\frac{n_1}{n'_1}\right)^2 \left(\frac{n_2}{n'_2}\right) = \frac{1}{n_g}$
$m > 0 \Rightarrow$ either image/object is virtual.

---

strong doublet - zonal aberration

Negative 'aplanats'
first surface object centered followed by aplanatic
$m = m_{\text{obj.cent.}} \cdot m_{apl.} = \left(\frac{n_1}{n'_1}\right)\left(\frac{n_2}{n'_2}\right)^2 = n_g$

---

pupil centered surface: $B = 0$
$\Rightarrow S_n$ that depends on B equals 0. (coma, astigmatism, distortion vanish)
SA, Petz. curvature remains.

Schmidt camera
$B = 0$, with an additional aspheric surface located at aperture stop)introduces only spherical).
Aspheric surface counteract SA, so only Petzval remains.
image surface is concentric with stop
–
but sphere introduces SA independent of λ, while that of plate varies with dispersion curve of glass. The spherochromatism (ϵ_y vs. ρ_y) looks cubic (F ($\lambda = 486.1nm$) curve looks like x^3, C ($\lambda = 656.3nm$) curve

looks like $-x^3$).
to counteract this, small amount of power is added to the corrector plate, to contribute axial color. (surface looks like, ray fans looks like pg. 195)

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refractive Schmidt
surface 1 aspheric, but no power. surface 2 concentric with stop ($B = 0$).
but system suffers axial color.
–
buried surface. (pg. 196)
thick glasses split into two with buried surface in between.
$n_d = n'_d$ on both sides of buried surface (so B still $= 0$ at surface 2), but different ν.

Field flatteners
thin $(-)$ lens at image plane ($y_a = 0 \quad \Rightarrow \quad$ SA, coma, astigmatism vanish). (small distortion)
purpose of this lens: decrease Petzval sum (to flatten the field).
(real field flattener is slightly away from image)
–
field flattener can also be placed at intermediate image (like field lens, but $(-)$ power)
eye relief ↑, field of view ↓
