6-1. We can answer the questions posed in this problem if we find the intensity point-spread function. From Eqs. (6-4) and (6-5), we know that the intensity point-spread function of an incoherent system is the squared magnitude of the (properly scaled) Fourier transform of the exit pupil illumination. The amplitude transmittance of the exit pupil in this case can be written

$$t_A(x,y) = \text{circ}\left(\frac{2r}{d}\right) \otimes [\delta(x - s/2, y) + \delta(x + s/2, y)]$$

where $r = \sqrt{x^2 + y^2}$. The Fourier transform of this expression is

$$\mathcal{F}\{t_A(x,y)\} = \pi\left(\frac{d}{2}\right)^2 2\frac{J_1(\pi d\rho)}{\pi d\rho} \times 2\cos(\pi s f_X),$$

where $\rho = \sqrt{f_X^2 + f_Y^2}$. Taking the squared magnitude of this expression, using the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, and introducing the scaling parameters appropriate for the optical Fourier transform, we obtain the following expression for the intensity point-spread function (under the assumption that the intensity of the wave at the exit pupil is unity):

$$I(u,v) = |h(u,v)|^2 = \frac{\pi^2 d^4}{16\lambda^2 z_i^2}\left[2\frac{J_1\left(\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}\right)}{\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}}\right]^2 \left[1 + \cos\left(\frac{2\pi s u}{\lambda z_i}\right)\right].$$

We can now answer the specific questions of the problem:

(a) The spatial frequency of the fringe is clearly given by

$$f_0 = \frac{s}{\lambda z_i}.$$

Note that the fringe frequency increases as the separation between the two apertures increases.

(b) The envelope of the fringe pattern is seen to be an Airy pattern of the form

$$E(u,v) = \left[2\frac{J_1\left(\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}\right)}{\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}}\right]^2,$$

where the scaling factor preceding the Airy pattern has been neglected.

chap 6:
thin lens: enter and exit at approx the same coordinates.
wavefront delay by $\propto$ thickness at that point $\Delta(x,y)$
suppose maximum distance (axis) Δ_0. then total phase delay suffer by wave at (x,y) passing through lens is
$\phi(x,y) = kn\Delta(x,y) + k(\Delta_0 - \Delta(x,y))$
multiplicative phase transformation $t_l(x,y) = e^{ik\Delta_0}e^{ik(n-1)\Delta(x,y)}$
then $U_l'(x,y) = t_l(x,y)U_l l(x,y)$

$\Delta(x,y) = \Delta_0 - R_1\left(1 - \sqrt{1 - \dfrac{x^2+y^2}{R_1^2}}\right) + R_2\left(1 - \sqrt{1 - \dfrac{x^2+y^2}{R_2^2}}\right)$
paraxial: $\Delta(x,y) = \Delta_0 - \dfrac{x^2+y^2}{2}\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$
paraxial: $t_l(x,y) = e^{ikn\Delta_0}e^{-ik(n-1)\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}$
$\dfrac{1}{f} \equiv (n-1)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right)$
ignore constant phase; $t_l(x,y) = e^{-i\frac{\pi}{\lambda f}(x^2+y^2)}$
$p(y_1,y_2) = L_0 + \dfrac{1}{2B}(Ay_1^2 - 2y_1y_2 + Dy_2^2)$
$L_0 = \sum_i n_i \Delta z_i$

---

planar input, normally incident, monochromatic plane wave of amp A: disturbance incident:

$$U_l(x,y) = At_A(x,y)$$

using Fresnel diffraction formula, with $z = f$:

$$U_f(u,v) = \frac{e^{i\frac{k}{2f}(u^2+v^2)}}{i\lambda f} \iint\limits_{-\infty}^{\infty} U_l(x,y) e^{-i\frac{2\pi}{\lambda f}(xu,yv)} dxdy$$

complex amp distribution of the field in focal plane is F-D pattern of field incident on lens. (with $f_X = \dfrac{u}{\lambda f}, f_Y = \dfrac{v}{\lambda f}$
Intensity is then

$$\begin{aligned}
I_f(u,v) &= \frac{A^2}{\lambda^2 f^2}\left| t_A(x,y) e^{-i\frac{2\pi}{\lambda f}(xu+yv)} dxdy \right|^2 \\
F_o(f_X,f_Y) &= \mathcal{F}\{At_A\} \qquad F_l(f_X,f_Y) = \mathcal{F}\{U_l\} \\
\text{drop constant phase} \qquad F_l(f_X,f_Y) &= F_o(f_X,f_Y) e^{-i\pi\lambda d(f_X^2+f_Y^2)} \\
U_f(u,v) &= \frac{e^{i\frac{k}{2f}(u^2+v^2)}}{i\lambda f} F_l\left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right) \\
&= \frac{e^{i\frac{k}{2f}\left(1-\frac{d}{f}\right)(u^2+v^2)}}{i\lambda f} F_o\left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right) \\
&= \frac{Ae^{i\frac{k}{2f}\left(1-\frac{d}{f}\right)(u^2+v^2)}}{i\lambda f} \iint\limits_{-\infty}^{\infty} t_A(\xi,\eta) e^{-i\frac{2\pi}{\lambda f}(\xi u+\eta v)} d\xi d\eta \\
\text{pupil centered at} \qquad &= \frac{Ae^{i\frac{k}{2f}\left(1-\frac{d}{f}\right)(u^2+v^2)}}{i\lambda f} \iint\limits_{-\infty}^{\infty} t_A(\xi,\eta) P\left(\xi + \frac{d}{f}u, \eta + \frac{d}{f}v\right) e^{-i\frac{2\pi}{\lambda f}(\xi u+\eta v)} d\xi d\eta
\end{aligned}$$

For transparency (distance d in front of f) behind the lens, the field distribution is

$$U_f(u,v) = \frac{Af}{i\lambda d^2} e^{i\frac{k}{2d}(u^2+v^2)} \mathcal{F}\left\{ t_A(\xi,\eta) P\left(\xi\frac{f}{d}, \eta\frac{f}{d}\right)\right\}_{f_X=\frac{u}{\lambda f}, f_Y=\frac{v}{\lambda f}}$$

---

paraxial approx:

$$\begin{aligned}
U_o(\xi,\eta) &= \left(\frac{Af}{d} P\left(\xi\frac{f}{d}, \eta\frac{f}{d}\right) e^{-i\frac{k}{2d}(\xi^2+\eta^2)}\right) t_A(\xi,\eta) \\
\Rightarrow U_f(u,v) &= \frac{Ae^{i\frac{k}{2d}(u^2+v^2)}}{i\lambda d} \frac{f}{d} \iint\limits_{-\infty}^{\infty} t_A(\xi,\eta) P\left(\xi\frac{f}{d}, \eta\frac{f}{d}\right) e^{-i\frac{2\pi}{\lambda d}(\xi u+\eta v)} d\xi d\eta
\end{aligned}$$

$U_o(\xi,,\eta)$ - complex field immediately behind the object. $U_i(u,v)$ - field distribution distance z_2 behind the lens.
$U_i(u,v) = \int_{-\infty}^{\infty} h(u,v;\xi,\eta) U_o(\xi,\eta) d\xi d\eta$
high quality images: U_i as similar as possible to U_o.
$h(u,v;\xi,\eta) \approx K\delta(u - M\xi, v - M\eta)$
where K complex constant, M magnification.

Let object be a δ function at coordinates (ξ, η)
the incident on lens appear as spherical wave from (ξ, η)
paraxial: $U_l(x, y) = \frac{1}{i\lambda z_1} e^{i\frac{k}{2z_1}[(x-\xi)^2+y-\eta)^2]}$
passage through lens: $U'_l(x, y) = U_l(x, y)P(x, y)e^{-i\frac{k}{2f}(x^2=y^2)}$
then propagate over z_2

$$h(u, v; \xi, \eta) = \frac{1}{i\lambda z_2} \iint\limits_{-\infty}^{\infty} U'_l(x, y) e^{i\frac{k}{2z_2}[(u-x)^2+v-y)^2]} dxdy$$

$$= \frac{1}{\lambda^2 z_1 z_2} e^{i\frac{k}{2z_2}(u^2+v^2)} e^{i\frac{k}{2z_1}(\xi^2+\eta^2)} \cdot \iint\limits_{-\infty}^{\infty} P(x, y) e^{i\frac{k}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)(x^2+y^2)} \cdot e^{-ik\left(\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{v}{z_2}\right)y\right)} dxdy$$

lens law (eliminate x,y phase): $\frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f} = 0$
(u, v) phase ignored under these 2 conditions:
1) it is the intensity distribution in the image plane that is of interest (phase goes away for intensity).
2) image field distribution is of interest, but image is measured on spherical surface, centered at point where the optical axis pierces the thin lens, and of radius z_2.
(ξ, η) phase neglected under 3 diff conditions:
1) object exists on surface of sphere of radius z_1 centered on the point (def. z_l) where optical axis pierces lens.
2) object illuminated by spherical wave converging towards z_l.
3) phase changes by amount $\ll 1$ radian within region of object.

$$h(u, v; \xi, \eta) \approx \frac{1}{\lambda^2 z_1 z_2} \iint\limits_{-\infty}^{\infty} P(x, y) \cdot e^{-ik\left(\left(\frac{\xi}{z_1}+\frac{u}{z_2}\right)x+\left(\frac{\eta}{z_1}+\frac{v}{z_2}\right)y\right)} dxdy$$

$$M = -\frac{z_2}{z_1}$$

$$h(u, v; \xi, \eta) \approx \frac{1}{\lambda^2 z_1 z_2} \iint\limits_{-\infty}^{\infty} P(x, y) \cdot e^{-i\frac{2\pi}{\lambda z_2}((u-M\xi)x+(v-M\eta)y)} dxdy$$

---

ideal image
according 2 geometrical optics, image and object would be related by
$U_i(u, v) = \frac{1}{|M|} U_o\left(\frac{u}{M}, \frac{v}{M}\right)$
Transfer function that produce this result is $h(u, v; \xi, \eta) = \frac{1}{|M|}\delta\left(\xi - \frac{u}{M}, \eta - \frac{v}{M}\right)$
Let $\tilde{\xi} = M\xi, \tilde{\eta} = M\eta, \tilde{h} = \frac{1}{|M|}h, \tilde{x} = \frac{x}{\lambda z_2}, \tilde{y} = \frac{y}{\lambda z_2}$
object image -relationship becomes

$$U_i(u, v) = \iint\limits_{-\infty}^{\infty} \tilde{h}(u - \tilde{\xi}, v - \tilde{\eta}) \left(\frac{1}{|M|} U_o\left(\frac{\tilde{\xi}}{M}, \frac{\tilde{\eta}}{M}\right)\right)$$

$$= \tilde{h}(u, v) \otimes U_g(u, v)$$

$$\text{where} \qquad U_g(u,v) = \frac{1}{|M|} U_o\left(\frac{u}{M}, \frac{v}{M}\right)$$

$$\tilde{h} = \int\int_{-\infty}^{\infty} P(\lambda z_2 \tilde{x}, \lambda z_2 \tilde{y}) e^{-2\pi i(u\tilde{x}+v\tilde{y})} d\tilde{x} d\tilde{y}$$

$$\tilde{h} = \frac{1}{|M|} h \qquad \tilde{x} = \frac{x}{\lambda z_2} \qquad \tilde{y} = \frac{y}{\lambda z_2}$$

ideal image produced by a diffraction-limited optical system (sys free from aberration) is a scaled and inverted version of object.
effect of diffraction is to convolve tat ideal image with the Fraunhofer diffraction pattern of the lens pupil.

ABCD matrix
The field distribution is

$$U_2(x,y) = \frac{e^{ikL_0}}{i\lambda B} e^{i\frac{\pi D}{\lambda B}(x^2+y^2)} \mathcal{F}\left\{U_1(\xi,\eta) e^{i\frac{\pi A}{\lambda B}(\xi^2+\eta^2)}\right\}_{f_X=\frac{x}{\lambda B}, f_Y=\frac{y}{\lambda b}}$$

Frequency analysis of optical imaging system
partial coherence.
coherent imaging system is linear in complex amplitude.
incoherence imaging system is linear in intensity, ir of system is squared magnitude of the amp ir.
polychromatic wave $u(P,t)$. suppress all $(+)$ freq components of Fourier spectrum, and double its $(-)$ freq components:
$u_-(P,t) = U(P,t)e^{-2\pi i\overline{\nu}t}$
where $\overline{\nu}$ is the mean freq of optical wave.

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imaging
$\tilde{\xi} = M\xi, \tilde{\eta} = M\eta$
$h(u,v) = \frac{A}{\lambda z_i} \int\int_{-\infty}^{\infty} P(x,y) e^{-i\frac{2\pi}{\lambda z_i}[(u-\tilde{\xi})x+(v-\tilde{\eta})y]} dxdy \propto \mathcal{F}_{f_X,f_Y\to u,v} P(\lambda z_i f_X, \lambda z_i f_Y)$ where z_i is distance from exit pupil 2 image.
image of perfect imaging system: $U_g(\tilde{\xi},\tilde{\eta}) = \frac{1}{|M|} U_o\left(\frac{\tilde{\xi}}{M}, \frac{\tilde{\eta}}{M}\right)$
Then field at image is convolution: $U_i(u,v) = U_g * h$

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narrowband: $U_i(u,v;t) = \int\int_{-\infty}^{\infty} h(u-\tilde{\xi}, v-\tilde{\eta}) U_g(\tilde{\xi},\tilde{\eta}; t-\tau) d\tilde{\xi} d\tilde{\eta}$
where τ is time delay of propagation from
image intensity is the time average of the instantaneous intensity:
$I_i(u,v) = \int\int_{-\infty}^{\infty} d\tilde{\xi}_1 d\tilde{\eta}_1 \int\int_{-\infty}^{\infty} d\tilde{\xi}_2 d\tilde{\eta}_2 h(u-\tilde{\xi}_1, v-\tilde{\eta}_1) h^*(u-\tilde{\xi}_2, v-\tilde{\eta}_2) \cdot \left\langle U_g(\tilde{\xi}_1,\tilde{\eta}_1; t-\tau_1) U_g^*(\tilde{\xi}_2,\tilde{\eta}_2; t-\tau_2)\right\rangle$
ir is nonzero over small region about ideal image point.
$\tau_1 - \tau_2 \approx 0$ for narrowband. drop τ_1, τ_2 in integral.
coherent, incoherent
incoherent: $\theta_s \geq \theta_o + \theta_p$
coherent: $\theta_s \ll \theta_p$

$$I_i(u,v) = \iint_{-\infty}^{\infty} d\tilde{\xi}_1 d\tilde{\eta}_1 \iint_{-\infty}^{\infty} d\tilde{\xi}_2 d\tilde{\eta}_2 h(u-\tilde{\xi}_1, v-\tilde{\eta}_1) h^*(u-\tilde{\xi}_2, v-\tilde{\eta}_2) J_g(\tilde{\xi}_1,\tilde{\eta}_1;\tilde{\xi}_2,\tilde{\eta}_2)$$

$$J_g(\tilde{\xi}_1,\tilde{\eta}_1;\tilde{\xi}_2,\tilde{\eta}_2) = \left\langle U_g(\tilde{\xi}_1,\tilde{\eta}_1;t) U_g^*(\tilde{\xi}_2,\tilde{\eta}_2;t) \right\rangle$$

$$\text{perfectly coherent} \qquad = \left\langle U_g(\tilde{\xi}_1,\tilde{\eta}_1) U_g^*(\tilde{\xi}_2,\tilde{\eta}_2) \right\rangle$$

$$\text{perfectly coherent} \qquad \Rightarrow I_i(u,v) = \left| \iint_{-\infty}^{\infty} h(u-\tilde{\xi}, v-\tilde{\eta}) U_g(\tilde{\xi},\tilde{\eta}) \right|^2$$

$$\text{perfectly coherent} \qquad U_i(u,v) = \iint_{-\infty}^{\infty} h(u-\tilde{\xi}, v-\tilde{\eta}) U_g(\tilde{\xi},\tilde{\eta}) d\tilde{\xi} d\tilde{\eta}$$

$$\text{perfectly incoherent} \qquad J_g(\tilde{\xi}_1,\tilde{\eta}_1;\tilde{\xi}_2,\tilde{\eta}_2) = \kappa I_g(\tilde{\xi}_1,\tilde{\eta}_1) \delta(\tilde{\xi}_1-\tilde{\xi}_2, \tilde{\eta}_1-\tilde{\eta}_2)$$

$$\Rightarrow I_i(u,v) = \kappa \iint_{-\infty}^{\infty} |h(u-\tilde{\xi}, v-\tilde{\eta})|^2 I_g(\tilde{\xi},\tilde{\eta}) d\tilde{\xi} d\tilde{\eta}$$

$$G_g(f_X,f_Y) = \mathcal{F}\{U_g(u,v)\}$$

$$G_i(f_X,f_Y) = \mathcal{F}\{U_i(u,v)\}$$

$$H(f_X,f_Y) = \mathcal{F}\{h(u,v)\}$$

$$\Rightarrow \qquad G_i(f_X,f_Y) = H(f_X,f_Y) G_g(f_X,f_Y)$$

$$H(f_X,f_Y) = \mathcal{F}\left\{ \frac{A}{\lambda z_i} \mathcal{F}\{P(x,y)\}_{f_X=\frac{u}{\lambda z_i}, f_Y=\frac{v}{\lambda z_i}} \right\}$$

$$= (A\lambda z_i) P(-\lambda z_i f_X, -\lambda z_i f_Y)$$

$$\text{convention} \qquad = P(\lambda z_i f_X, \lambda z_i f_Y)$$

$$\text{cutoff frequency} \qquad f_0 = \frac{w}{\lambda z_i}$$

$$\text{normalized frequency spectra of } I_g \qquad \mathcal{G}_g(f_X,f_Y) = \frac{\mathcal{F}\{I_g(u,v)\}_{f_X,f_Y}}{\mathcal{F}\{I_g(u,v)\}_{f_X=f_Y=0}}$$

$$\mathcal{G}_i(f_X,f_Y) = \text{same}$$

$$\text{normalized transfer function} \qquad \mathcal{H}(f_X,f_Y) = \frac{\mathcal{F}\{|h(u,v)|^2\}_{f_X,f_Y}}{\mathcal{F}\{|h(u,v)|^2\}_{0,0}}$$

$$\mathcal{G}_i(f_X,f_Y) = \mathcal{H}(f_X,f_Y) \mathcal{G}_g(f_X,f_Y)$$

OTF (normalized transfer function → Rayleigh's theorem → substitute to make it symmetric) is normalized autocorrelation function of the amplitude transfer function -

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properties of OTF
$\mathcal{H}(0,0) = 1$
$\mathcal{H}(-f_X,-f_Y) = \mathcal{H}^*(f_X,f_Y)$
$|\mathcal{H}(f_X,f_Y)| \leq \mathcal{H}(0,0)|$

OTF examples

$$\mathcal{H}(f_X, f_Y) = \frac{\text{area of overlap}}{\text{total area}}$$

$$\text{square aperture} \qquad = \Lambda\left(\frac{f_X}{2f_0}\right)\Lambda\left(\frac{f_Y}{2f_0}\right)$$

$$f_0 = \frac{w}{\lambda z_i}$$

$$\text{circular aperture} \qquad \mathcal{H}(\rho) = \begin{cases} \frac{2}{\pi}\left(\cos^{-1}\left(\frac{\rho}{2\rho_0}\right) - \frac{\rho}{2\rho_0}\sqrt{1-\left(\frac{\rho}{2\rho_0}\right)^2}\right) & \rho \le 2\rho_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\rho_0 = \frac{w}{\lambda z_i}$$

---

complex amplitude transmittance of imaginary phase-shifting plate (generalized pupil function)
$\mathcal{P}(x,y) = P(x,y)e^{ikW(x,y)}$
if system free of aberration, exit pupil would be filled by perfect spherical wave converging toward ideal image point.
Gaussian reference sphere.
ir is F.T. of pupil function
amplitude transfer function is

$$H(f_X, f_Y) = P(\lambda z_i f_X, \lambda z_i f_Y)e^{ikW(\lambda z_i f_X, \lambda z_i f_Y)}$$

$$\text{area of overlap}\mathcal{A}(f_X, f_Y) = \text{area of } P\left(x - \frac{\lambda z_i f_X}{2}, y - \frac{\lambda z_i f_Y}{2}\right), P\left(x + \frac{\lambda z_i f_X}{2}, y + \frac{\lambda z_i f_Y}{2}\right)$$

$$\text{aberrations} \qquad \mathcal{H}(f_X, f_Y) =$$

aberration never increase MTF (modulus of OTF).
severe aberration can reduce high-frequency portions of OTF to such extent that effective cut off is much lower than the diffraction-limited cutoff.
path-length error

$$W(x,y) = -\frac{1}{2}\left(\frac{1}{z_a} - \frac{1}{z_i}\right)(x^2+y^2)$$

$$W_m = -\frac{1}{2}\left(\frac{1}{z_a} - \frac{1}{z_i}\right)w^2 \Rightarrow W(x,y) \qquad = W_m\frac{x^2+y^2}{w^2}$$

when aberrations of any kind are severe, the geometrical optics predictions of the intensity point-spread function may be fourier transformed to yield a good approx to the OTF of system.

$$\text{incoherent} \qquad I_i = |h|^2 \otimes I_g = |h|^2 \otimes |U_g|^2$$

$$\text{coherent} \qquad I_i = |h \otimes U_g|^2$$

$$\text{autocorrelation} \qquad X(f_X, f_Y) \star X(f_X, f_Y) = \int\int_{-\infty}^{\infty} X(p,q)X^*(p-f_X, q-f_Y)dpdq$$

$$\text{incoherent} \qquad \mathcal{F}\{I_i\} = [H \star H][G_g \star G_g]$$

$$\text{coherent} \qquad I_i = HG_g \star HG_g$$

where G_g is spectrum of U_g and H is amplitude transfer function.

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psf = ift of otf (normalize not necessary?)
The point spread function (PSF) describes the response of an imaging system to a point source or point object. A more general term for the PSF is a system's impulse response.
When aberrations of any kind are severe, the geometrical optics predictions of the intensity psf may be F.T. to yield a good approx to the OTF of the system. (diffraction play negligible role when severe aberrations are present)

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coherent vs. incoherent
object amp in coherent case, object intensity in incoherent case.
intensity point-spread function of an incoherent system is the squared magnitude of the (properly scaled) Fourier transform of the exit pupil illumination.

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abbrevation
ir - impulse response.
F.T. - fourier transform
OTF - optical transfer function

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2 point resolution
Rayleigh criterion of resolution. 2 incoherent point source by diffraction limited system with circular pupil. Airy first zero has peak of second Airy.
$\delta = 0.61 \frac{\lambda}{\sin \theta}$

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speckle effect
problem of speckle effect observed with highly coherent illumination.
size of speckles roughly the size of a resolution cell on the image (or object).

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super-resolution, or bandwidth extrapolation.
in absence of noise.
Theorem 1: The 2D F.T. of a spatially bounded function is an analytic function in the (f_X, f_Y) plane.
Theorem 2: If an ana function in the (f_X, f_Y) plane is known exactly in an arbitrarily small (buy finite) region of that plane, then the entire function can be found (uniquely) by means of analytic continuation.